Abrahamse's Theorem for matrix-valued symbols and subnormal Toeplitz completions

Raúl E. Curto, In Sung Hwang and Woo Young Lee

Abstract. This paper deals with subnormality of Toeplitz operators with matrix-valued symbols and, in particular, with an appropriate reformulation of Halmos's Problem 5: Which subnormal Toeplitz operators with matrix-valued symbols are either normal or analytic? In 1976, M. Abrahamse showed that if $\varphi \in L^\infty$ is such that φ or $\overline{\varphi}$ is of bounded type and if T_φ is subnormal, then T_φ is either normal or analytic. In this paper we establish a matrix-valued version of Abrahamse's Theorem and then apply this result to solve the following Toeplitz completion problem: Find the unspecified Toeplitz entries of the partial block Toeplitz matrix

$$A := \begin{bmatrix} T_{\overline{b}_\alpha} & ? \\ ? & T_{\overline{b}_\beta} \end{bmatrix} \quad (\alpha, \beta \in \mathbb{D})$$

so that A becomes subnormal, where b_λ is a Blaschke factor of the form $b_\lambda(z) := \frac{z-\lambda}{1-\overline{\lambda}z}$ $(\lambda \in \mathbb{D})$.

1. Introduction

This paper focuses on subnormality for Toeplitz operators with matrix-valued symbols and more precisely, the case of Toeplitz operators with matrix-valued bounded type symbols. In this paper we give an appropriate generalization of Abrahamse's Theorem to the case of matrix-valued symbols and apply this generalization to solve a subnormal Toeplitz completion problem.

To describe our results in more detail, we first need to review a few essential facts about (block) Toeplitz operators, and for that we will use [BS], [Do1], [Do2], [GGK], [Ni] and [Pe]. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *hyponormal* if its self-commutator $[T^*, T] := T^*T - TT^*$ is positive (semi-definite), and *subnormal* if there exists a normal operator N on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $\mathcal{H}$ is invariant under N and $N|_{\mathcal{H}} = T$. Let $\mathbb{T} \equiv \partial\,\mathbb{D}$ be the unit circle in the complex plane. Let $L^2 \equiv L^2(\mathbb{T})$ be the set of all square-integrable measurable functions on $\mathbb{T}$ and let $H^2 \equiv H^2(\mathbb{T})$ be the corresponding Hardy space. Let $H^\infty \equiv H^\infty(\mathbb{T}) := L^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$, that is, H^∞ is the set of bounded analytic functions on $\mathbb{D}$. Given $\varphi \in L^\infty$, the Toeplitz operator T_φ and the Hankel operator H_φ are defined by

$$T_\varphi g := P(\varphi g) \quad \text{and} \quad H_\varphi g := JP^\perp(\varphi g) \qquad (g \in H^2),$$

MSC(2010): Primary 47B20, 47B35, 46J15, 15A83; Secondary 30H10, 47A20
Keywords: Block Toeplitz operators; subnormal; Abrahamse's Theorem; bounded type functions; subnormal completion problems
The work of the first named author was partially supported by NSF Grant DMS-0801168. The work of second named author was supported by Basic Science Research Program through NRF funded by the Ministry of Education, Science and Technology (2011-0022577). The work of the third author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No.2012-0000939).

where P and $P^\perp$ denote the orthogonal projections that map from L^2 onto H^2 and $(H^2)^\perp$, respectively, and where J denotes the unitary operator on L^2 defined by $J(f)(z) = \overline{z}f(\overline{z})$. In 1988, the hyponormality of T_φ was completely characterized in terms of its symbol via Cowen's Theorem [Co3].

Cowen's Theorem. ([Co3], [NT]) *For each $\varphi \in L^\infty$, let*

$$\mathcal{E}(\varphi) \equiv \{k \in H^\infty : \ ||k||_\infty \le 1 \ \textit{and} \ \varphi - k\overline{\varphi} \in H^\infty\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

This elegant and useful theorem has been used in [CuL1], [CuL2], [FL], [Gu1], [Gu2], [GS], [HKL1], [HKL2], [HL1], [HL2], [HL3], [Le], [NT] and [Zhu], which have been devoted to the study of hyponormality for Toeplitz operators on H^2. When one studies the hyponormality (also, normality and subnormality) of the Toeplitz operator T_φ one may, without loss of generality, assume that $\varphi(0) = 0$; this is because hyponormality is invariant under translation by scalars.

We now recall that a function $\varphi \in L^\infty$ is said to be of *bounded type* (or in the Nevanlinna class) if there are analytic functions $\psi_1, \psi_2 \in H^\infty(\mathbb{D})$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)} \quad \text{for almost all } z \in \mathbb{T}.$$

If $\varphi \in L^\infty$, we write

$$\varphi_+ \equiv P\varphi \in H^2 \quad \text{and} \quad \varphi_- \equiv \overline{P^\perp \varphi} \in zH^2.$$

Let BMO denote the set of functions of bounded mean oscillation in L^1. Then $L^\infty \subseteq BMO \subseteq L^2$. It is well-known that if $f \in L^2$, then H_f is bounded on H^2 whenever $P^\perp f \in BMO$ (cf. [Pe]). If $\varphi \in L^\infty$, then $\overline{\varphi_-}, \overline{\varphi_+} \in BMO$, so that $H_{\overline{\varphi_-}}$ and $H_{\overline{\varphi_+}}$ are well understood. It is well known [Ab, Lemma 3] that if $\varphi \in L^\infty$ then

$$\varphi \text{ is of bounded type} \iff \ker H_\varphi \neq \{0\}\,. \tag{1}$$

Assume now that both φ and $\overline{\varphi}$ are of bounded type. Since $T_{\overline{z}} H_\psi = H_\psi T_z$ for all $\psi \in L^\infty$, it follows from Beurling's Theorem that $\ker H_{\overline{\varphi_-}} = \theta_0 H^2$ and $\ker H_{\overline{\varphi_+}} = \theta_+ H^2$ for some inner functions θ_0, θ_+. We thus have $b := \overline{\varphi_-}\theta_0 \in H^2$, and hence we can write

$$\varphi_- = \theta_0 \overline{b}, \text{ and similarly } \varphi_+ = \theta_+ \overline{a} \text{ for some } a \in H^2. \tag{2}$$

In the factorization (2), we will always assume that θ_0 and b are coprime and θ_+ and a are coprime. In particular, if T_φ is hyponormal and $\varphi \notin H^\infty$, and since

$$[T_\varphi^*, T_\varphi] = H^*_{\overline{\varphi}} H_{\overline{\varphi}} - H^*_\varphi H_\varphi = H^*_{\overline{\varphi_+}} H_{\overline{\varphi_+}} - H^*_{\overline{\varphi_-}} H_{\overline{\varphi_-}},$$

it follows that $||H_{\overline{\varphi_+}} f|| \ge ||H_{\overline{\varphi_-}} f||$ for all $f \in H^2$, and hence

$$\theta_+ H^2 = \ker H_{\overline{\varphi_+}} \subseteq \ker H_{\overline{\varphi_-}} = \theta_0 H^2,$$

which implies that θ_0 divides θ_+, i.e., $\theta_+ = \theta_0 \theta_1$ for some inner function θ_1. We write, for an inner function θ,

$$\mathcal{H}_\theta := H^2 \ominus \theta\, H^2.$$

Note that if $f = \theta \overline{a} \in L^2$, then $f \in H^2$ if and only if $a \in \mathcal{H}_{z\theta}$; in particular, if $f(0) = 0$ then $a \in \mathcal{H}_\theta$. Thus, if $\varphi = \overline{\varphi_-} + \varphi_+ \in L^\infty$ is such that φ and $\overline{\varphi}$ are of bounded type and T_φ is hyponormal, then we can write

$$\varphi_+ = \theta_0 \theta_1 \overline{a} \quad \text{and} \quad \varphi_- = \theta_0 \overline{b}, \qquad \text{where } a \in \mathcal{H}_{z\theta_0\theta_1} \text{ and } b \in \mathcal{H}_{\theta_0}: \tag{3}$$

in this case, $\theta_0 \theta_1 \overline{a}$ and $\theta_0 \overline{b}$ are called *coprime factorizations* of φ_+ and φ_-, respectively. By Kronecker's Lemma [Ni, p. 183], if $f \in H^\infty$ then $\overline{f}$ is a rational function if and only if rank $H_{\overline{f}} < \infty$, which implies that

$$\overline{f} \text{ is rational} \iff f = \theta \overline{b} \ \text{ with a finite Blaschke product } \theta. \tag{4}$$

We now introduce the notion of block Toeplitz operators. For a Hilbert space $\mathcal{X}$, let $L^2_{\mathcal{X}} \equiv L^2_{\mathcal{X}}(\mathbb{T})$ be the Hilbert space of $\mathcal{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$ and

let $H^2_{\mathcal{X}} \equiv H^2_{\mathcal{X}}(\mathbb{T})$ be the corresponding Hardy space. We observe that $L^2_{\mathbb{C}^n} = L^2 \otimes \mathbb{C}^n$ and $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$. If Φ is a matrix-valued function in $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbb{T})$ $(= L^\infty \otimes M_n)$ then $T_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ denotes the block Toeplitz operator with symbol Φ defined by

$$T_\Phi F := P_n(\Phi F) \quad \text{for } F \in H^2_{\mathbb{C}^n},$$

where P_n is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$. A block Hankel operator with symbol $\Phi \in L^\infty_{M_n}$ is the operator $H_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ defined by

$$H_\Phi F := J_n P_n^\perp(\Phi F) \quad \text{for } F \in H^2_{\mathbb{C}^n},$$

where $P_n^\perp$ is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $(H^2_{\mathbb{C}^n})^\perp$, J_n denotes the unitary operator on $L^2_{\mathbb{C}^n}$ given by $J_n(F)(z) := \overline{z} I_n F(\overline{z})$ for $F \in L^2_{\mathbb{C}^n}$, and where I_n is the $n \times n$ identity matrix. For $\Phi \in L^\infty_{M_n}$, we write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}).$$

For $\Phi \in L^\infty_{M_n}$, we also write

$$\Phi_+ := P_n \Phi \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \big(P_n^\perp \Phi\big)^* \in H^2_{M_n}.$$

Thus we can write $\Phi = \Phi_-^* + \Phi_+$. However, it will be often convenient to permit the constant term for Φ_-. Hence, if there is no confusion we may assume that Φ_- shares the constant term with Φ_+: in this case, $\Phi(0) = \Phi_+(0) + \Phi_-(0)^*$.

A matrix-valued function $\Theta \in H^\infty_{M_{n\times m}}$ $(= H^\infty \otimes M_{n\times m})$ is called *inner* if $\Theta(z)^*\Theta(z) = I_m$ for almost all $z \in \mathbb{T}$. The following basic relations can be easily derived:

$$T_\Phi^* = T_{\Phi^*}, \quad H_\Phi^* = H_{\widetilde{\Phi}} \quad (\Phi \in L^\infty_{M_n});$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi \quad (\Phi, \Psi \in L^\infty_{M_n}); \tag{5}$$

$$H_\Phi T_\Psi = H_{\Phi\Psi}, \quad H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi \quad (\Phi \in L^\infty_{M_n}, \Psi \in H^\infty_{M_n}); \tag{6}$$

For a matrix-valued function $\Phi = [\phi_{ij}] \in L^\infty_{M_n}$, we say that Φ is of *bounded type* if each entry ϕ_{ij} is of bounded type and that Φ is *rational* if each entry ϕ_{ij} is a rational function.

In 2006, Gu, Hendricks and Rutherford [GHR] characterized the hyponormality of block Toeplitz operators in terms of their symbols. In particular they showed that if T_Φ is a hyponormal block Toeplitz operator on $H^2_{\mathbb{C}^n}$, then Φ is normal, i.e., $\Phi^*\Phi = \Phi\Phi^*$. Their characterization for hyponormality of block Toeplitz operators resembles Cowen's Theorem except for an additional condition – the normality condition of the symbol.

Lemma 1.1. (**Hyponormality of Block Toeplitz Operators**) (Gu-Hendricks-Rutherford [GHR]) For each $\Phi \in L^\infty_{M_n}$, let

$$\mathcal{E}(\Phi) := \Big\{K \in H^\infty_{M_n} : \ ||K||_\infty \le 1 \ \text{ and } \ \Phi - K\Phi^* \in H^\infty_{M_n}\Big\}.$$

Then T_Φ is hyponormal if and only if Φ is normal and $\mathcal{E}(\Phi)$ is nonempty.

In [GHR], the normality of block Toeplitz operator T_Φ was also characterized in terms of the symbol Φ, under a "determinant" assumption on the symbol Φ.

Lemma 1.2. (**Normality of Block Toeplitz Operators**) (Gu-Hendricks-Rutherford [GHR]) Let $\Phi \equiv \Phi_+ + \Phi_-^*$ be normal. If $\det \Phi_+$ is not identically zero then

$$T_\Phi \text{ is normal} \Longleftrightarrow \Phi_+ - \Phi_+(0) = \big(\Phi_- - \Phi_-(0)\big)\, U \ \text{ for some constant unitary matrix } U. \tag{7}$$

On the other hand, M. Abrahamse [Ab, Lemma 6] showed that if T_φ is hyponormal, if $\varphi \notin H^\infty$, and if φ or $\overline{\varphi}$ is of bounded type then both φ and $\overline{\varphi}$ are of bounded type. However, by contrast to the scalar case, Φ^* may not be of bounded type even though T_Φ is hyponormal, $\Phi \notin H^\infty_{M_n}$ and Φ is of bounded type. But we have a one-way implication (see [GHR, Corollary 3.5 and Remark 3.6]):

$$T_\Phi \text{ is hyponormal and } \Phi^* \text{ is of bounded type} \implies \Phi \text{ is of bounded type.} \tag{8}$$

For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$ $(m \le n)$. We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are said to be *coprime* if they are both left and right coprime. We remark that if $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero then any left inner divisor Δ of Φ is square, i.e., $\Delta \in H^2_{M_n}$. If $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero then we say that $\Delta \in H^2_{M_n}$ is a *right inner divisor* of Φ if $\widetilde{\Delta}$ is a left inner divisor of $\widetilde{\Phi}$.

The following lemma will be useful in the sequel.

Lemma 1.3. ([GHR]) For $\Phi \in L^\infty_{M_n}$, the following statements are equivalent:

(i) Φ is of bounded type;
(ii) $\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$ for some square inner matrix function Θ;
(iii) $\Phi = A\Theta^*$, where $A \in H^\infty_{M_n}$ and A and Θ are right coprime.

For an inner matrix function $\Theta \in H^\infty_{M_n}$, write

$$\mathcal{H}_\Theta := \left(\Theta H^2_{\mathbb{C}^n}\right)^\perp \equiv H^2_{\mathbb{C}^n} \ominus \Theta H^2_{\mathbb{C}^n}.$$

Suppose $\Phi = [\varphi_{ij}] \in L^\infty_{M_n}$ is such that Φ^* is of bounded type. Then we may write $\varphi_{ij} = \theta_{ij}\overline{b}_{ij}$, where θ_{ij} is an inner function and θ_{ij} and b_{ij} are coprime. Thus if θ is the least common multiple of θ_{ij}'s (i.e., the θ_{ij} divide θ and if they divide an inner function θ' then θ in turn divides θ'), then we can write

$$\Phi = [\varphi_{ij}] = [\theta_{ij}\overline{b}_{ij}] = [\theta\overline{a}_{ij}] = \Theta A^* \quad (\Theta = \theta I_n,\ A \in H^2_{M_n}). \tag{9}$$

We note that the representation (9) is "minimal," in the sense that if ωI_n (ω is inner) is a common inner divisor of Θ and A, then ω is constant. Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Then in view of (9) we can write

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*,$$

where $\Theta_i = \theta_i I_n$ with an inner function θ_i for $i = 1, 2$ and $A, B \in H^2_{M_n}$. In particular, if $\Phi \in L^\infty_{M_n}$ is rational then the θ_i are chosen as finite Blaschke products as we observed in (4).

In this paper we consider the subnormality of block Toeplitz operators and in particular, the matrix-valued version of Halmos's Problem 5: Which subnormal Toeplitz operators with matrix-valued symbols are either normal or analytic? In 1976, M. Abrahamse showed that if $\varphi \in L^\infty$ is such that φ or $\overline{\varphi}$ is of bounded type, if T_φ is hyponormal, and if $\ker [T^*_\varphi, T_\varphi]$ is invariant under T_φ then T_φ is either normal or analytic. The purpose of this paper is to establish a matrix-valued version of Abrahamse's Theorem and then apply this result to solve a Toeplitz completion problem. In Section 2 we make a brief sketch on Halmos's Problem 5 and the earlier results. Section 3 is devoted to get an Abrahamse's Theorem for matrix-valued symbols. In Section 4, using our extension of Abrahamse's Theorem for matrix-valued symbols, we solve the following 'Toeplitz completion" problem: find the unspecified Toeplitz entries of the partial block Toeplitz matrix

$$A := \begin{bmatrix} T_{\overline{b}_\alpha} & ? \\ ? & T_{\overline{b}_\beta} \end{bmatrix} \quad (\alpha, \beta \in \mathbb{D})$$

so that A becomes subnormal, where b_λ is a Blaschke factor of the form $b_\lambda(z) := \frac{z-\lambda}{1-\overline{\lambda}z}$ $(\lambda \in \mathbb{D})$.

2. Halmos's Problem 5 and Abrahamse's Theorem

In 1970, P.R. Halmos posed the following problem, listed as Problem 5 in his lecture "Ten problems in Hilbert space" [Hal1], [Hal2]:

Is every subnormal Toeplitz operator either normal or analytic?

A Toeplitz operator T_φ is called *analytic* if $\varphi \in H^\infty$. Any analytic Toeplitz operator is easily seen to be subnormal: indeed, $T_\varphi h = P(\varphi h) = \varphi h = M_\varphi h$ for $h \in H^2$, where M_φ is the normal operator of multiplication by φ on L^2. The question is natural because the two classes, the normal and analytic Toeplitz operators, are fairly well understood and are subnormal. In 1984, Halmos's Problem 5 was answered in the negative by C. Cowen and J. Long [CoL]. However, unfortunately, Cowen and Long's construction does not provide an intrinsic connection between subnormality and the theory of Toeplitz operators. Until now researchers have been unable to characterize subnormal Toeplitz operators in terms of their symbols.

We would like to reformulate Halmos's Problem 5 as follows:

Halmos's Problem 5 reformulated. *Which Toeplitz operators are subnormal?*

The most interesting partial answer to Halmos's Problem 5 was given by M. Abrahamse [Ab]. M. Abrahamse gave a general sufficient condition for the answer to Halmos's Problem 5 to be affirmative. Abrahamse's Theorem can be then stated as:

Abrahamse's Theorem ([Ab, Theorem]). *Let $\varphi \in L^\infty$ be such that φ or $\overline{\varphi}$ is of bounded type. If T_φ is hyponormal and* $\ker [T_\varphi^*, T_\varphi]$ *is invariant under T_φ then T_φ is normal or analytic.*

Consequently, if $\varphi \in L^\infty$ is such that φ or $\overline{\varphi}$ is of bounded type, then every subnormal Toeplitz operator must be either normal or analytic.

We say that a block Toeplitz operator T_Φ is *analytic* if $\Phi \in H^\infty_{M_n}$. Evidently, any analytic block Toeplitz operator with a normal symbol is subnormal because the multiplication operator M_Φ is a normal extension of T_Φ. As a first inquiry in the above reformulation of Halmos's Problem 5 the following question can be raised:

Is Abrahamse's Theorem valid for block Toeplitz operators?

In [CHL2, Theorem 3.5], the authors gave a matrix-valued version of Abrahamse's Theorem. As a corollary the following result was shown:

Theorem 2.1. ([CHL2, Corollary 3.9]). Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is a matrix-valued rational function. Then in view of (9) and (4), we may write

$$\Phi_- = B^* \Theta, \tag{10}$$

where $\Theta := \theta I_n$ with a finite Blaschke product θ. Assume that B and Θ are (left) coprime. If T_Φ is subnormal then T_Φ is either normal or analytic.

Note that in the coprime factorization (10) of Φ_-, Θ is a *diagonal-constant* inner function, i.e., a diagonal inner function, constant along the diagonal. This assumption seems to be too rigid. To see this, we consider the following example.

Example 2.2. Let $b_\alpha := \frac{z-\alpha}{1-\overline{\alpha}z}$ $(\alpha \in \mathbb{D})$, let θ be an inner function which is coprime with b_α and b_β $(\alpha \neq \beta)$ and consider the following matrix-valued function

$$\Phi := \begin{bmatrix} \overline{\theta} & \overline{\theta b_\alpha} + c\theta b_\beta \\ \overline{b_\beta} + c\theta^2 b_\alpha & \overline{\theta} \end{bmatrix}, \quad \text{where } c \in \mathbb{R} \text{ with } c \geq \left|\left| \begin{bmatrix} \theta & b_\beta \\ \theta b_\alpha & \theta \end{bmatrix} \right|\right|_\infty. \tag{11}$$

Then

$$\Phi_+ = c \begin{bmatrix} 0 & \theta b_\beta \\ \theta^2 b_\alpha & 0 \end{bmatrix} \quad \text{and} \quad \Phi_- = \begin{bmatrix} \theta & b_\beta \\ \theta b_\alpha & \theta \end{bmatrix}.$$

A straightforward calculation shows that $\Phi^*\Phi = \Phi\Phi^*$. If $K := \frac{1}{c} \begin{bmatrix} \theta & b_\beta \\ \theta b_\alpha & \theta \end{bmatrix}$, then

$$||K||_\infty \leq 1 \quad \text{and} \quad \Phi_-^* = K \Phi_+^*,$$

which implies that by Lemma 1.1, T_Φ is hyponormal. But a direct calculation shows that T_Φ is not normal. On the other hand, we observe

$$\widetilde{\Phi_-} = \begin{bmatrix} \widetilde{\theta} & \widetilde{\theta}\widetilde{b}_\alpha \\ \widetilde{b}_\beta & \widetilde{\theta} \end{bmatrix},$$

so that

$$\widetilde{\Phi_-}^* \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \overline{\widetilde{\theta}} & \overline{\widetilde{b}_\beta} \\ \overline{\widetilde{\theta}\widetilde{b}_\alpha} & \overline{\widetilde{\theta}} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \in H^2_{\mathbb{C}^2} \iff \begin{cases} \overline{\widetilde{\theta}} f + \overline{\widetilde{b}_\beta} g \in H^2 \\ \overline{\widetilde{\theta}\widetilde{b}_\alpha} f + \overline{\widetilde{\theta}} g \in H^2 \end{cases},$$

which implies (by using the assumption that θ is coprime with b_α and b_β),

$$f \in \widetilde{\theta}\widetilde{b}_\alpha H^2 \quad \text{and} \quad g \in \widetilde{\theta}\widetilde{b}_\beta H^2 .$$

Thus we have

$$\ker H_{\widetilde{\Phi_-}^*} = \begin{bmatrix} \widetilde{\theta}\widetilde{b}_\alpha & 0 \\ 0 & \widetilde{\theta}\widetilde{b}_\beta \end{bmatrix} H^2_{\mathbb{C}^2},$$

so that, by Lemma 1.3, we can get

$$\widetilde{\Phi_-} = \begin{bmatrix} \widetilde{\theta} & \widetilde{\theta}\widetilde{b}_\alpha \\ \widetilde{b}_\beta & \widetilde{\theta} \end{bmatrix} = \begin{bmatrix} \widetilde{\theta}\widetilde{b}_\alpha & 0 \\ 0 & \widetilde{\theta}\widetilde{b}_\beta \end{bmatrix} \begin{bmatrix} \widetilde{b}_\alpha & \widetilde{\theta} \\ 1 & \widetilde{b}_\beta \end{bmatrix}^* \equiv \widetilde{\Theta}\widetilde{B}^* \quad \text{(right coprime factorization)},$$

where

$$\widetilde{\Theta} := \begin{bmatrix} \widetilde{\theta}\widetilde{b}_\alpha & 0 \\ 0 & \widetilde{\theta}\widetilde{b}_\beta \end{bmatrix} \quad \text{and} \quad \widetilde{B} := \begin{bmatrix} \widetilde{b}_\alpha & \widetilde{\theta} \\ 1 & \widetilde{b}_\beta \end{bmatrix}.$$

Hence we get

$$\Phi_- = B^*\Theta = \begin{bmatrix} b_\alpha & 1 \\ \theta & b_\beta \end{bmatrix}^* \begin{bmatrix} \theta b_\alpha & 0 \\ 0 & \theta b_\beta \end{bmatrix} \quad \text{(left coprime factorization)}.$$

But since $\Theta \equiv \begin{bmatrix} \theta b_\alpha & 0 \\ 0 & \theta b_\beta \end{bmatrix}$ is not diagonal-constant we cannot apply Theorem 2.1 to determine whether or not T_Φ is subnormal. However, as we will see in the sequel, we can conclude (using Theorem 3.8 below) that T_Φ is not subnormal. $\square$

3. Abrahamse's Theorem for matrix-valued symbols

Recall the representation (9), and for $\Psi \in L^\infty_{M_n}$ such that Ψ^* is of bounded type, write $\Psi = \Theta_2 B^* = B^*\Theta_2$. Let Ω be the greatest common left inner divisor of B and Θ_2. Then $B = \Omega B_\ell$ and $\Theta_2 = \Omega\Omega_2$ for some $B_\ell \in H^2_{M_n}$ and some inner matrix Ω_2. Therefore we can write

$$\Psi = B_\ell^*\Omega_2, \quad \text{where } B_\ell \text{ and } \Omega_2 \text{ are left coprime:} \tag{12}$$

in this case, $B_\ell^*\Omega_2$ is called a *left coprime factorization* of Ψ. Similarly,

$$\Psi = \Delta_2 B_r^*, \quad \text{where } B_r \text{ and } \Delta_2 \text{ are right coprime:} \tag{13}$$

in this case, $\Delta_2 B_r^*$ is called a *right coprime factorization* of Ψ.

To prove our main results, we need several auxiliary lemmas.
We begin with:

Lemma 3.1. (a) Let $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Then in view of (9), we may write

$$\Phi_+ = A^*\Theta_1 \quad \text{and} \quad \Phi_- = B^*\Theta_2,$$

where $\Theta_i := \theta_i I_n$ with an inner function θ_i $(i = 1, 2)$. If T_Φ is hyponormal, then Θ_2 is a right inner divisor of Θ_1.

(b) In view of (13), $\Phi \in L^\infty_{M_n}$ may be written as

$$\Phi_+ = \Delta_1 A_r^* \text{ (right coprime factorization)} \quad \text{and} \quad \Phi_- = \Delta_2 B_r^* \text{ (right coprime factorization)}.$$

If T_Φ is hyponormal, then Δ_2 is a left inner divisor of Δ_1.

Proof. See [CHL2, Lemmas 3.1 and 3.2]. □

In the sequel, when we consider the symbol $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$, which is such that Φ and Φ^* are of bounded type and for which T_Φ is hyponormal, we will, in view of Lemma 3.1, assume that

$$\Phi_+ = A^*\Omega_1\Omega_2 \quad \text{and} \quad \Phi_- = B_\ell^*\Omega_2 \text{ (left coprime factorization)}, \tag{14}$$

where $\Omega_1\Omega_2 = \Theta = \theta I_n$. We also note that $\Omega_2\Omega_1 = \Theta$: indeed, if $\Omega_1\Omega_2 = \Theta = \theta I_n$, then $(\overline{\theta} I_n\Omega_1)\Omega_2 = I_n$, so that $\Omega_1(\overline{\theta} I_n\Omega_2) = I_n$, which implies that $(\overline{\theta} I_n\Omega_2)\Omega_1 = I_n$, and hence $\Omega_2\Omega_1 = \theta I_n = \Theta$.

We recall the inner-outer factorization of vector-valued functions. If D and E are Hilbert spaces and if F is a function with values in $\mathcal{B}(E, D)$ such that $F(\cdot)e \in H^2_D(\mathbb{T})$ for each $e \in E$, then F is called a strong H^2-function. The strong H^2-function F is called an *inner* function if $F(\cdot)$ is an isometric operator from D into E. Write $\mathcal{P}_E$ for the set of all polynomials with values in E, i.e., $p(\zeta) = \sum_{k=0}^n \widehat{p}(k)\zeta^k$, $\widehat{p}(k) \in E$. Then the function $Fp = \sum_{k=0}^n F\widehat{p}(k)z^k$ belongs to $H^2_D(\mathbb{T})$. The strong H^2-function F is called *outer* if

$$\text{cl}\, F \cdot \mathcal{P}_E = H^2_D(\mathbb{T}).$$

Note that every $F \in H^2_{M_n}$ is a strong H^2-function. We then have an analogue of the scalar Inner-Outer Factorization Theorem.

Inner-Outer Factorization. (cf. [Ni]) Every strong H^2-function F with values in $\mathcal{B}(E, D)$ can be expressed in the form

$$F = F^i F^e,$$

where F^e is an outer function with values in $\mathcal{B}(E, D')$ and F^i is an inner function with values in $\mathcal{B}(D', D)$ for some Hilbert space D'.

We introduce a key idea which provides a connection between left coprime-ness and right coprime-ness.

Definition 3.2. If $\Delta \in H^\infty_{M_n}$ is an inner function, we define

$$D(\Delta) := \text{GCD}\left\{\theta I_n : \ \theta \text{ is inner and } \Delta \text{ is a (left) inner divisor of } \theta I_n\right\},$$

where GCD $(\cdot)$ denotes the greatest common inner divisor.

Lemma 3.3. If $\Delta \in H^\infty_{M_n}$ is an inner function then

$$D(\Delta) = \delta I_n \quad \text{for some inner funtion } \delta. \tag{15}$$

Proof. Let $\Delta \in H^\infty_{M_n}$ be inner. Then since Δ^* is evidently of bounded type, we can write, in view of (9),

$$\Delta = \Theta A^* \quad \text{with } \Theta \equiv \theta I_n \text{ for an inner function } \theta \text{ and } A \in H^2_{M_n}.$$

But since Δ is inner it follows that $A^*A = I_n$, so that $\Delta A = \Theta$. This says that Δ is a left inner divisor of θI_n. Thus $D(\Delta)$ always exists for each inner function $\Delta \in H^\infty_{M_n}$. For (15), we observe that for any index set I,

$$D(\Delta)H^2_{\mathbb{C}^n} = \bigvee_{i \in I} (\theta_i I_n)H^2_{\mathbb{C}^n} = \bigoplus_{j=1}^n \bigvee_{i\in I} \theta_i H^2 = \bigoplus_{j=1}^n \text{GCD}\left\{\theta_i : i \in I\right\}H^2, \tag{16}$$

which implies that $D(\Delta) = \delta I_n$ with $\delta := \text{GCD}\left\{\theta_i : i \in I\right\}$, giving (15). □

Note that $D(\Delta)$ is unique up to a diagonal-constant inner function of the form $e^{i\xi} I_n$.

If one of two inner functions is diagonal-constant then the "left" coprime-ness and the "right" coprime-ness between them coincide.

Lemma 3.4. Let $\Delta \in H^\infty_{M_n}$ be inner and $\Theta := \theta I_n$ for some inner function θ. Then the following are equivalent:

(a) Θ and Δ are left coprime;
(b) Θ and Δ are right coprime;
(c) Θ and $D(\Delta)$ are coprime.

Proof. We first prove the equivalence (b) $\Leftrightarrow$ (c).

(c) $\Rightarrow$ (b): Evident.

(b) $\Rightarrow$ (c): If Δ is a diagonal-constant inner function then this is trivial. Thus we suppose that Δ is not diagonal-constant. Write

$$D(\Delta) := \delta I_n \quad \text{and} \quad D(\Delta) = \Delta\Delta_0 = \Delta_0\Delta \quad \text{for a nonconstant inner function } \Delta_0.$$

Suppose Θ and $D(\Delta)$ are not coprime. Then θ and δ are not coprime. Put

$$\omega := \text{GCD}\,(\theta,\ \delta) \quad \text{and} \quad \Omega := \omega I_n.$$

Thus

$$\Theta = \Omega\Theta_1 \quad \text{and} \quad D(\Delta) = \Omega\Delta_1,$$

where $\Theta_1 = \theta_1 I_n$ and $\Delta_1 = \delta_1 I_n$ for some inner functions θ_1 and δ_1. Then

$$\theta I_n = \Omega\Theta_1 \quad \text{and} \quad \Delta\Delta_0 = \delta I_n = \Omega\Delta_1. \tag{17}$$

If $\delta = \omega$ then δI_n is an inner divisor of θI_n, so that, evidently, Θ and Δ are not right coprime. We now suppose $\delta \neq \omega$. We then claim that

$$\Delta \text{ and } \Omega \text{ are not right coprime.} \tag{18}$$

For (18), we assume to the contrary that Δ and Ω are right coprime. Since by (17),

$$\delta_1 I_n = \Delta\Delta_0\overline{\omega}I_n = \Delta(\overline{\omega}I_n\Delta_0),$$

it follows that ωI_n is an inner divisor of Δ_0, so that $\overline{\omega}I_n\Delta_0$ is inner. Consequently,

$$(\delta\overline{\omega})I_n = \delta_1 I_n = \Delta(\overline{\omega}I_n\Delta_0),$$

which contradicts to the definition of $D(\Delta)$. This proves (18). But since $\Theta = \Omega\Theta_1 = \Theta_1\Omega$, it follows that Θ and Δ are not right coprime.

(a) $\Leftrightarrow$ (c). Since $\widetilde{D(\Delta)} = D(\widetilde{\Delta})$, it follows from the equivalence (b) $\Leftrightarrow$ (c) that

$$\begin{aligned} \Theta \text{ and } D(\Delta) \text{ are coprime} &\iff \widetilde{\Theta} \text{ and } \widetilde{D(\Delta)} \text{ are coprime} \\ &\iff \widetilde{\Theta} \text{ and } \widetilde{\Delta} \text{ are right coprime} \\ &\iff \Theta \text{ and } \Delta \text{ are left coprime.} \end{aligned}$$

This completes the proof. $\square$

Lemma 3.5. Let $A \in H^2_{M_n}$ be such that $\det A$ is not identically zero and $\Theta := \theta I_n$ for some inner function θ. Then the following are equivalent:

(a) Θ and A are left coprime;
(b) Θ and A are right coprime.

Proof. Since $\det A$ is not identically zero, the left and the right inner divisors of A are square. Thus we have the following inner-outer factorizations of A of the form

$$A = A^i\, A^e = B^e\, B^i\,,$$

where $A^i, B^i \in H^2_{M_n}$ are inner and $A^e, B^e \in H^2_{M_n}$ are outer. We will show that

$$D(A^i) = D(B^i). \tag{19}$$

Write

$$D(B^i) = B^i\Delta_0 \quad \text{for some inner function } \Delta_0.$$

Then we have

$$\begin{aligned} D(B^i)H^2_{\mathbb{C}^n} &= D(B^i)\Big[\mathrm{cl}\, B^e\mathcal{P}_{\mathbb{C}^n}\Big] = \mathrm{cl}\, B^e\Big[D(B^i)\mathcal{P}_{\mathbb{C}^n}\Big] = \mathrm{cl}\, B^e B^i\Big[\Delta_0\mathcal{P}_{\mathbb{C}^n}\Big] \\ &= \mathrm{cl}\, A^i A^e\Big[\Delta_0\mathcal{P}_{\mathbb{C}^n}\Big] = A^i\Big[\mathrm{cl}\, A^e\Delta_0\mathcal{P}_{\mathbb{C}^n}\Big] \subseteq A^i H^2_{\mathbb{C}^n}, \end{aligned}$$

which proves that $D(B^i)H^2_{\mathbb{C}^n} \subseteq A^i H^2_{\mathbb{C}^n}$, so that A^i is a left inner divisor of $D(B^i)$. Thus by the definition of $D(A^i)$, $D(A^i)$ is a (left) inner divisor of $D(B^i)$. Similarly, we can show that $D(B^i)$ is an inner divisor of $D(A^i)$, and hence $D(A^i) = D(B^i)$. Thus by Lemma 3.4, we have

$$\begin{aligned} \Theta \text{ and } A \text{ are left coprime} &\Longleftrightarrow \Theta \text{ and } D(A^i) \text{ are coprime} \\ &\Longleftrightarrow \Theta \text{ and } D(B^i) \text{ are coprime} \\ &\Longleftrightarrow \Theta \text{ and } A \text{ are right coprime.} \end{aligned}$$

□

In Lemma 3.5, if θ is given as a finite Blaschke product then the "determinant" assumption may be dropped.

Lemma 3.6. Let $A \in H^2_{M_n}$ and $\Theta := \theta I_n$ for a finite Blaschke product θ. Then the following are equivalent:

(a) Θ and A are left coprime;
(b) Θ and A are right coprime;
(c) $A(\alpha)$ is invertible for each zero α of θ.

Proof. See [CHL2, Lemma 3.3]. □

Lemma 3.7. Let $A \in H^2_{M_n}$ and Θ be a diagonal inner function whose diagonal entries are nonconstant. If $Af = 0$ for each $f \in \mathcal{H}_\Theta$, then $A = 0$.

Proof. Write $A \equiv [a_{ij}]_{1\le i,j\le n}$ ($a_{ij} \in H^2$) and $\Theta \equiv \mathrm{diag}(\theta_1, \cdots, \theta_n)$ (where θ_j is a nonconstant inner function for each $j = 1, \cdots, n$). Suppose $Af = 0$ for each $f \in \mathcal{H}_\Theta$. Choose an outer function h_j in $\mathcal{H}_{\theta_j}$ which is invertible in H^∞ (cf. [CHL2, Lemma 3.4]). For each $j = 1, \cdots, n$, define $g_j := (0, \cdots, 0, h_j, 0, \cdots, 0)^t$ (where h_j is the j-th component). Clearly, $g_j \in \mathcal{H}_\Theta$. Thus by assumption, $Ag_j = 0$ for each j, so that $a_{ij}h_j = 0$ for each $i, j = 1, \cdots, n$. But since h_j is invertible, $a_{ij} = 0$ for each $i, j = 1, \cdots, n$, i.e., $A = 0$. □

We are now ready to prove the main result of this paper.

Theorem 3.8. (Abrahamse's Theorem for matrix-valued symbols, Version I) Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type and $\det \Phi_+$ and $\det \Phi_-$ are not identically zero. Then in view of (12), we may write

$$\Phi_- = B^*\Theta \quad \text{(left coprime factorization)}.$$

Assume that Θ is a diagonal inner matrix function (which is not necessarily diagonal-constant) and that Θ has a nonconstant diagonal-constant inner divisor $\Omega \equiv \omega I_n$ (ω inner) such that Ω and $\Theta\Omega^*$ are coprime. If

(i) T_Φ is hyponormal; and
(ii) $\ker\, [T^*_\Phi, T_\Phi]$ is invariant under T_Φ ,

then T_Φ is either normal or analytic. Hence, in particular, if T_Φ is subnormal then it is either normal or analytic.

Proof. For notational convenience, we let $\Theta_2 := \Theta$. In view of Lemma 3.1(a), we may write

$$\Phi_+ = \Theta_0 \Theta_2 A^*,$$

where $\Theta_0\Theta_2 = \theta I_n$ with an inner function θ and $A \in H^2_{M_n}$. If Θ_2 is constant then $\Phi_- \in M_n$, so that T_Φ is analytic. Suppose that Θ_2 is nonconstant and $\Theta_2 := \Omega\Delta$, where $\Omega \equiv \omega I_n$ with a nonconstant inner function ω, and Ω and Δ are coprime.

We split the proof into six steps: each step is significant as a separate mathematical statement.

STEP 1: We first claim that

$$\Theta_0 H^2_{\mathbb{C}^n} \ \subseteq\ \ker\,[T^*_\Phi, T_\Phi]. \tag{20}$$

Indeed, the inclusion (20) follows from a slight extension of [CHL2, Theorem 3.5], in which Θ_2 is a diagonal inner function of the form $\Theta_2 = \theta_2 I_n$. In fact, a careful analysis for the proof of [CHL2, STEP 1 of the proof of Theorem 3.5] shows that the proof does not employ the diagonal-*constant*-ness of Θ_2, but uses only the diagonal-constant-ness of $\Theta_0\Theta_2$.

STEP 2: We also argue that if $K \in \mathcal{E}(\Phi)$, then

$$\text{cl ran}\, H_{A\Theta_2^*} \subseteq \ker\,(I - T_{\widetilde{K}} T^*_{\widetilde{K}}). \tag{21}$$

To see this, we observe that if $K \in \mathcal{E}(\Phi)$ then by (6),

$$[T^*_\Phi, T_\Phi] = H^*_{\Phi^*_+} H_{\Phi^*_+} - H^*_{K\Phi^*_+} H_{K\Phi^*_+} = H^*_{\Phi^*_+}(I - T_{\widetilde{K}} T^*_{\widetilde{K}}) H_{\Phi^*_+}\,, \tag{22}$$

so that

$$\ker\,[T^*_\Phi, T_\Phi] = \ker\,(I - T_{\widetilde{K}} T^*_{\widetilde{K}}) H_{\Phi^*_+}\,.$$

Thus by (20),

$$\{0\} = (I - T_{\widetilde{K}} T^*_{\widetilde{K}}) H_{A\Theta_2^*\Theta_0^*}(\Theta_0 H^2_{\mathbb{C}^n}) = (I - T_{\widetilde{K}} T^*_{\widetilde{K}}) H_{A\Theta_2^*}(H^2_{\mathbb{C}^n})\,,$$

giving (21).

We note that STEP 1 and STEP 2 hold with no restriction on $\Theta \equiv \Theta_2$.

STEP 3: We claim that

$$\Omega \text{ and } \Theta_0 \text{ are (right) coprime.} \tag{23}$$

To see this we assume to the contrary that Ω and Θ_0 are not coprime. Since $\Theta_2\Theta_0 \equiv \Omega\Delta\Theta_0$ and Ω are diagonal-constant, it follows that $\Delta\Theta_0$ is diagonal-constant. Also there exists an inner function Δ' such that $\Delta\Delta' = D(\Delta)$. Thus we can write

$$\Theta_2\Theta_0 = \Omega\Delta\Theta_0 = \Omega\Delta\Delta'\Gamma$$

where $\Gamma := \gamma I_n$ for some inner function γ. Since by assumption, Ω and Δ are coprime it follows from Lemma 3.4 that Ω and $\Delta\Delta'$ are coprime. Therefore we have

$$\Omega' := \text{GCD}\,\{\Omega,\ \Theta_0\} \equiv \omega' I_n,$$

where $\omega' = \text{GCD}\{\omega, \gamma\}$ is not constant. Thus we can write

$$\Theta_0 = \Omega'\Theta_0' = \Theta_0'\Omega' \quad \text{and} \quad \Theta_2 = \Omega'\Theta_2' = \Theta_2'\Omega' \tag{24}$$

for some diagonal inner functions Θ_0', Θ_2'. Then since $\Theta_2\Theta_0' = \Omega'\Theta_2'\Theta_0' = \Omega'\Theta_0'\Theta_2' = \Theta_0\Theta_2'$, it follows from (20) that

$$\Theta_2\Theta_0' H^2_{\mathbb{C}^n} \subseteq \Theta_0 H^2_{\mathbb{C}^n} \subseteq \ker\,[T^*_\Phi, T_\Phi]. \tag{25}$$

Note that

$$\Theta_2\Theta_0' = \Omega\Delta\Theta_0' = \Omega\Delta\Delta'(\gamma\overline{\omega'})I_n \quad (\gamma\overline{\omega'} \in H^2).$$

Thus since $\Theta_2\Theta_0'$ is diagonal-constant and hence, $\Theta_2^* B\Theta_2\Theta_0' \in H^2_{M_n}$, it follows that

$$H_{\Phi^*_-}(\Theta_2\Theta_0' H^2_{\mathbb{C}^n}) = H_{\Theta_2^* B}(\Theta_2\Theta_0' H^2_{\mathbb{C}^n}) = 0\,.$$

Thus by (25), we have

$$H_{\Phi^*_+}(\Theta_2\Theta_0' H^2_{\mathbb{C}^n}) = H_{A\Theta_2^*\Theta_0^*}(\Theta_2\Theta_0' H^2_{\mathbb{C}^n}) = \{0\}, \quad \text{so that} \ \ H_{A\Omega'^*}(H^2_{\mathbb{C}^n}) = \{0\}.$$

Thus we must have that $G \equiv A\Omega'^* \in H^2_{M_n}$. Then we can write

$$\Phi_+ = \Theta_0\Theta_2 A^* = \Omega'\Theta_0'\Theta_2'\Omega' A^* = \Omega'\Theta_0'\Theta_2' G^*,$$

which leads to a contradiction because the representation $\Phi_+ = \Theta_0\Theta_2 A^*$ is in "minimal" form in view of (9). This proves (23).

STEP 4: We claim that

$$A \text{ and } \Omega \text{ are left coprime.} \tag{26}$$

Indeed, by assumption B and Θ_2 are left coprime, so we can see that B and Ω are left coprime. Since $\det \Phi_-$ is not identically zero and hence, $\det B$ is not either, it follows from Lemma 3.5 that B and Ω are right coprime. Thus by Lemma 3.1(b), we can write

$$\Phi_+ = \Theta_0\Theta_2 A^* = \Omega\Delta_1 A_r^*\,,$$

where A_r and $\Omega\Delta_1$ are right coprime. In particular, since by (23), Ω and Θ_0 are right coprime, A and Ω are right coprime. Since by assumption, $\det \Phi_+$ is not identically zero and hence, $\det A$ is not either, it follows again from Lemma 3.5 that A and Ω are left coprime. This proves (26).

STEP 5: We now claim that

$$\mathcal{E}(\Phi) \text{ contains an inner function } K. \tag{27}$$

We first observe that $A\Theta_2^* = A\Omega^*\Delta^* = \Omega^* A \Delta^*$, so that

$$\text{cl ran}\, H_{A\Theta_2^*} = \left(\ker H_{\widetilde{\Delta}^*\widetilde{A}\widetilde{\Omega}^*}\right)^{\perp}.$$

Since by (26), A and Ω are left coprime (so that $\widetilde{A}$ and $\widetilde{\Omega}$ are right coprime), it follows that

$$\begin{aligned} f \in \ker H_{\widetilde{\Delta}^*\widetilde{A}\widetilde{\Omega}^*} &\Longrightarrow \widetilde{\Delta}^*\widetilde{A}\widetilde{\Omega}^* f \in H^2_{\mathbb{C}^n} \\ &\Longrightarrow \widetilde{A}\widetilde{\Omega}^* f \in \widetilde{\Delta} H^2_{\mathbb{C}^n} \subseteq H^2_{\mathbb{C}^n} \\ &\Longrightarrow f \in \ker H_{\widetilde{A}\widetilde{\Omega}^*} \\ &\Longrightarrow f \in \widetilde{\Omega} H^2_{\mathbb{C}^n} \quad \text{(by Lemma 1.3)}, \end{aligned}$$

which implies that

$$\ker H_{\widetilde{\Delta}^*\widetilde{A}\widetilde{\Omega}^*} \subseteq \widetilde{\Omega} H^2_{\mathbb{C}^n}\,,$$

and hence,

$$\mathcal{H}_{\widetilde{\Omega}} \subseteq \text{cl ran}\, H_{A\Theta_2^*}\,.$$

Thus by (21),

$$\mathcal{H}_{\widetilde{\Omega}} \subseteq \ker\,(I - T_{\widetilde{K}} T_{\widetilde{K}}^*). \tag{28}$$

We thus have

$$F = T_{\widetilde{K}} T_{\widetilde{K}}^* F \quad \text{for each } F \in \mathcal{H}_{\widetilde{\Omega}}\,. \tag{29}$$

But since $||\widetilde{K}||_\infty = ||K||_\infty \leq 1$, it follows from a direct calculation that

$$||P_n(\widetilde{K}^* F)||_2 = ||\widetilde{K}^* F||_2,$$

which implies $\widetilde{K}^* F \in H^2_{\mathbb{C}^n}$. Therefore by (29), $(I - \widetilde{K}\widetilde{K}^*)F = 0$ for each $F \in \mathcal{H}_{\widetilde{\Omega}}$. Thus by Lemma 3.7, $K^*K = I$, which proves (27).

STEP 6: We finally claim that

$$T_\Phi \text{ is normal.}$$

To see this, we first observe that if $K \in \mathcal{E}(\Phi)$ is inner then it follows from (28) that

$$\mathcal{H}_{\widetilde{\Omega}} \subseteq \ker\,(I - T_{\widetilde{K}} T_{\widetilde{K}}^*) = \ker H_{\widetilde{K}^*}^* H_{\widetilde{K}^*} = \ker H_{\widetilde{K}^*}\,. \tag{30}$$

Write $K := [k_{ij}]_{1 \leq i,j \leq n} \in H^\infty_{M_n}$. It thus follows that for each $i, j = 1, 2, \cdots, n$,

$$k_{ij}(\overline{z})h \in H^2 \quad \text{for an invertible function } h \in \mathcal{H}_{\widetilde{\omega}}.$$

Therefore each k_{ij} is constant and hence, K is constant. Therefore by (22), $[T_\Phi^*, T_\Phi] = 0$, i.e., T_Φ is normal. This completes the proof. □

In Theorem 3.8, if Θ has a nonconstant diagonal-constant inner divisor of the form ωI_n with a Blaschke factor ω, then we can strengthen Theorem 3.8 by dropping the "determinant" assumption.

Corollary 3.9. Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type. Then in view of (12), we may write

$$\Phi_- = B^*\Theta \quad \text{(left coprime factorization)},$$

where Θ is a diagonal inner matrix function. Assume that Θ has a nonconstant diagonal-constant inner divisor $\Omega \equiv \omega I_n$ with a finite Blaschke product ω such that Ω and $\Theta\Omega^*$ are coprime. If

(i) T_Φ is hyponormal; and
(ii) $\ker\,[T_\Phi^*, T_\Phi]$ is invariant under T_Φ,

then T_Φ is either normal or analytic. Hence, in particular, if T_Φ is subnormal then it is either normal or analytic.

Proof. If we put $\Omega := \omega I_n$ with a finite Blaschke product ω, then we may use Lemma 3.6 in place of Lemma 3.5. Thus we can drop the "determinant" condition in Theorem 3.8 because Theorem 3.8 employs the determinant condition only for the equivalence of the left coprime-ness and the right coprime-ness between Ω and some $D \in H^2_{M_n}$. $\square$

In Corollary 3.9, if Θ is diagonal-constant then we may take $\Theta = \Omega$, and hence $\Theta\Omega^* = I$, so that Θ and $\Theta\Omega^*$ are trivially coprime. Thus if Θ is diagonal-constant then Corollary 3.9 reduces to Theorem 2.1.

Example 3.10. (Example 2.2 Revisited) We take a chance to reconsider the function given in (11):

$$\Phi := \begin{bmatrix} \overline{\theta} & \overline{\theta b_\alpha} + c\theta b_\beta \\ \overline{b_\beta} + c\theta^2 b_\alpha & \overline{\theta} \end{bmatrix}, \quad \text{where } c \in \mathbb{R} \text{ with } c \geq \left|\left| \begin{bmatrix} \theta & b_\beta \\ \theta b_\alpha & \theta \end{bmatrix} \right|\right|_\infty .$$

In Section 2 we have shown that T_Φ is hyponormal, but not normal. On the other hand, we know that

$$\Phi_- = B^*\Theta = \begin{bmatrix} b_\alpha & 1 \\ \theta & b_\beta \end{bmatrix}^* \begin{bmatrix} \theta b_\alpha & 0 \\ 0 & \theta b_\beta \end{bmatrix} \quad \text{(left coprime factorization)},$$

But since

$$\Theta = \begin{bmatrix} \theta b_\alpha & 0 \\ 0 & \theta b_\beta \end{bmatrix} = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} \begin{bmatrix} b_\alpha & 0 \\ 0 & b_\beta \end{bmatrix}$$

and $\begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}$ and $\begin{bmatrix} b_\alpha & 0 \\ 0 & b_\beta \end{bmatrix}$ are coprime (since θ is coprime with b_α and b_β), it follows from Theorem 3.8 that T_Φ is not subnormal. $\square$

Remark 3.11. The assumption "Θ is diagonal" in Theorem 3.8 seems to be still somewhat rigid. A careful analysis of the proof of Theorem 3.8 shows that this assumption was used only in proving STEP 3 (and whence STEP 4). However, we did not directly employ the assumption "$\Theta \equiv \Theta_2$ is diagonal" in the proofs of STEP 5 and STEP 6; instead we used the statement in STEP 4. Also, we have already recognized that STEP 1 and STEP 2 hold with no restriction on Θ. Therefore if we make the assumption "A and Ω are left coprime" in Theorem 3.8, then Theorem 3.8 still holds for a general form of Θ. Moreover, if we assume that A and Θ are left coprime then we do not need an additional assumption that Ω and $\Theta\Omega^*$ are coprime because it was used only in the proof of STEP 3 (as an auxiliary lemma for STEP 4). Consequently, if we strengthen the left coprime-ness for the analytic part of the symbol then we can relax the restriction on Θ.

Therefore we get:

Corollary 3.12. (Abrahamse's Theorem for matrix-valued symbols, Version II) Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type and $\det \Phi_+$ and $\det \Phi_-$ are not identically zero. Then in view of (14), we may write

$$\Phi_+ = A^* \Theta_0 \Theta_2 \quad \text{and} \quad \Phi_- = B^* \Theta_2 \,,$$

where $\Theta_0 \Theta_2 = \theta I_n$ with an inner function θ. Assume that A, B and Θ_2 are left coprime and Θ_2 has a nonconstant diagonal-constant inner divisor $\Omega \equiv \omega I_n$ (ω inner). If

(i) T_Φ is hyponormal; and
(ii) $\ker [T_\Phi^*, T_\Phi]$ is invariant under T_Φ ,

then T_Φ is either normal or analytic. Hence, in particular, if T_Φ is subnormal then it is either normal or analytic.

Proof. This follows from Remark 3.11 and an analysis of the proof of Theorem 3.8. □

Remark 3.13. Observe that Corollary 3.12 is a substantive generalization of [CHL2, Theorem 3.5], in which Θ_2 is diagonal-constant.

4. A Subnormal Toeplitz Completion

Given a partially specified operator matrix with some known entries, the problem of finding suitable operators to complete the given partial operator matrix so that the resulting matrix satisfies certain given properties is called a *completion problem.* A *subnormal completion* of a partial operator matrix is a particular specification of the unspecified entries resulting in a subnormal operator. A *partial block Toeplitz matrix* is simply an $n \times n$ matrix some of whose entries are specified Toeplitz operators and whose remaining entries are unspecified. A *subnormal Toeplitz completion* of a partial block Toeplitz matrix is a subnormal completion whose unspecified entries are Toeplitz operators.

In [CHL1], the following subnormal Toeplitz completion problem was considered:

Problem A. Let U be the unilateral shift on H^2. Complete the unspecified Toeplitz entries of the partial block Toeplitz matrix $A := \left[\begin{smallmatrix} U^* & ? \\ ? & U^* \end{smallmatrix}\right]$ to make A subnormal.

The solution of Problem A given in [CHL1, Theorem 5.1] relies upon very intricate and long computations using the symbol involved. In this section, by employing our main result in Section 3, we provide a shorter and more insightful proof for the following problem which is a more general version of Problem A:

Problem B. Let b_λ be a Blaschke factor of the form $b_\lambda(z) := \frac{z-\lambda}{1-\overline{\lambda} z}$ ($\lambda \in \mathbb{D}$). Complete the unspecified Toeplitz entries of the partial block Toeplitz matrix

$$A := \begin{bmatrix} T_{\overline{b}_\alpha} & ? \\ ? & T_{\overline{b}_\beta} \end{bmatrix} \quad (\alpha, \beta \in \mathbb{D})$$

to make A subnormal.

To answer Problem B, we need:

Lemma 4.1. Let

$$\Phi_- = \begin{bmatrix} b_\alpha & \theta_1 \overline{b} \\ \theta_0 \overline{a} & b_\alpha \end{bmatrix} \quad (a \in \mathcal{H}_{z\theta_0},\ b \in \mathcal{H}_{z\theta_1} \text{ and } \theta_j \text{ inner } (j=0,1))$$

and $\ker H_{\Phi_-^*} = \Delta H^2_{\mathbb{C}^2}$.

(a) If $\theta_0 = b_\alpha^n \theta_0'$ ($n \geq 1$, $\theta_0'(\alpha) \neq 0$) and $\theta_1(\alpha) \neq 0$, then

$$\Delta = \begin{cases} \begin{bmatrix} b_\alpha \theta_1 & 0 \\ 0 & \theta_0 \end{bmatrix} & (n=1); \\ \frac{1}{\sqrt{|\gamma|^2+1}} \begin{bmatrix} b_\alpha \theta_1 & \gamma \theta_1 \\ -\overline{\gamma} \theta_0 & b_\alpha^{n-1} \theta_0' \end{bmatrix} & (n \geq 2) \quad \left(\gamma := -\frac{a(\alpha)}{\theta_1(\alpha)}\right). \end{cases}$$

(b) If $\theta_1 = b_\alpha^n \theta_1'$ ($n \geq 1$, $\theta_1'(\alpha) \neq 0$) and $\theta_0(\alpha) \neq 0$, then

$$\Delta = \begin{cases} \begin{bmatrix} \theta_1 & 0 \\ 0 & b_\alpha \theta_0 \end{bmatrix} & (n=1); \\ \frac{1}{\sqrt{|\gamma|^2+1}} \begin{bmatrix} b_\alpha^{n-1}\theta_1' & -\overline{\gamma}\theta_1 \\ \gamma\theta_0 & b_\alpha\theta_0 \end{bmatrix} & (n \geq 2) \quad \left(\gamma := -\frac{b(\alpha)}{\theta_0(\alpha)}\right). \end{cases}$$

(c) If $\theta_0(\alpha) \neq 0$ and $\theta_1(\alpha) \neq 0$, then

$$\Delta = \begin{bmatrix} b_\alpha \theta_1 & 0 \\ 0 & b_\alpha \theta_0 \end{bmatrix}.$$

(d) If $\theta_0 = b_\alpha \theta_0'$ and $\theta_1 = b_\alpha \theta_1'$ then

$$\Delta = \begin{cases} \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_0 \end{bmatrix} & \Big((ab)(\alpha) \neq (\theta_0'\theta_1')(\alpha)\Big); \\ \frac{1}{\sqrt{|\gamma|^2+1}} \begin{bmatrix} \theta_1 & \gamma\theta_1' \\ -\overline{\gamma}\theta_0 & \theta_0' \end{bmatrix} & \Big((ab)(\alpha) = (\theta_0'\theta_1')(\alpha)\Big) \quad \left(\gamma := -\frac{a(\alpha)}{\theta_1'(\alpha)}\right). \end{cases}$$

Proof. This follows from a slight variation of the proof of [CHL1, Lemmas 5.4, 5.5, and 5.6]. □

We are ready for:

Theorem 4.2. Let $\varphi, \psi \in L^\infty$ and consider

$$A := \begin{bmatrix} T_{\overline{b}_\alpha} & T_\varphi \\ T_\psi & T_{\overline{b}_\beta} \end{bmatrix} \quad (\alpha, \beta \in \mathbb{D}),$$

where b_λ is a Blaschke factor of the form $b_\lambda(z) := \frac{z-\lambda}{1-\overline{\lambda}z}$ ($\lambda \in \mathbb{D}$). The following statements are equivalent.

(a) A is normal.
(b) A is subnormal.
(c) A is 2-hyponormal.
(d) $\alpha = \beta$ and one of the following conditions holds:

1. $\varphi = e^{i\theta} b_\alpha + \zeta$ and $\psi = e^{i\omega}\varphi$ ($\zeta \in \mathbb{C}$; $\theta, \omega \in [0, 2\pi)$);
2. $\varphi = \mu\, \overline{b}_\alpha + e^{i\theta}\sqrt{1+|\mu|^2}\, b_\alpha + \zeta$ and $\psi = e^{i\,(\pi - 2\arg\mu)}\varphi$ ($\mu, \zeta \in \mathbb{C}$, $\mu \neq 0$, $|\mu| \neq 1$, $\theta \in [0, 2\pi)$),

except in the following special case:

$$\varphi_- = b_\alpha \theta_0' \overline{a} \text{ and } \psi_- = b_\alpha \theta_1' \overline{b} \text{ (coprime factorizations) with } (ab)(\alpha) = (\theta_0'\theta_1')(\alpha) \neq 0. \tag{31}$$

However, if we also know that $\varphi, \psi \in L^\infty$ are rational functions having the same number of poles then either (2) holds for $|\mu| = 1$ or

$$\varphi = e^{i\theta}\overline{b}_\alpha + 2e^{i\omega} b_\alpha + \zeta \quad \text{and} \quad \psi = e^{-2i\theta}\varphi \quad (\theta, \omega \in [0, 2\pi),\ \zeta \in \mathbb{C}):$$

in this case, $A + e^{-i\theta}\zeta$ is quasinormal.

As a straightforward consequence of Theorem 4.2, we obtain

Corollary 4.3. Let

$$A := \begin{bmatrix} U^* & U^* + 2U \\ U^* + 2U & U^* \end{bmatrix},$$

where $U \equiv T_z$ is the unilateral shift on H^2. Then A is a quasinormal (therefore subnormal) completion of $\left[\begin{smallmatrix} U^* & ? \\ ? & U^* \end{smallmatrix}\right]$, and A is not normal.

Proof of Theorem 4.2. Clearly (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c). Moreover, (d) $\Rightarrow$ (a) follows from a straightforward calculation.

(c) $\Rightarrow$ (d): Write

$$\Phi \equiv \begin{bmatrix} \overline{b}_\alpha & \varphi \\ \psi & \overline{b}_\beta \end{bmatrix} \equiv \Phi_-^* + \Phi_+ = \begin{bmatrix} b_\alpha & \psi_- \\ \varphi_- & b_\beta \end{bmatrix}^* + \begin{bmatrix} 0 & \varphi_+ \\ \psi_+ & 0 \end{bmatrix}$$

and assume that T_Φ is 2-hyponormal. Since $\ker[T^*, T]$ is invariant under T for every 2-hyponormal operator $T \in \mathcal{B}(\mathcal{H})$, we note that Theorem 3.8 holds for 2-hyponormal operators T_Φ under the same assumption on the symbol. We claim that

$$|\varphi| = |\psi|, \text{ and} \tag{32}$$

$$\Phi \text{ and } \Phi^* \text{ are of bounded type.} \tag{33}$$

Indeed, if T_Φ is hyponormal then Φ is normal, so that a straightforward calculation gives (32). Also, by Lemma 1.1 there exists a matrix function $K \equiv \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right] \in \mathcal{E}(\Phi)$, i.e., $||K||_\infty \le 1$ such that $\Phi - K\Phi^* \in H^\infty_{M_2}$, i.e.,

$$\begin{bmatrix} \overline{b}_\alpha & \overline{\varphi_-} \\ \overline{\psi_-} & \overline{b}_\beta \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \begin{bmatrix} 0 & \overline{\psi_+} \\ \overline{\varphi_+} & 0 \end{bmatrix} \in H^2_{M_2}, \tag{34}$$

which implies that

$$H_{\overline{b}_\alpha} = H_{k_2\overline{\varphi_+}} = H_{\overline{\varphi_+}} T_{k_2} \quad \text{and} \quad H_{\overline{b}_\beta} = H_{k_3\overline{\psi_+}} = H_{\overline{\psi_+}} T_{k_3}.$$

If $\overline{\varphi_+}$ is not of bounded type then $\ker H_{\overline{\varphi_+}} = \{0\}$, so that $k_2 = 0$, a contradiction; and if $\overline{\psi_+}$ is not of bounded type then $\ker H_{\overline{\psi_+}} = \{0\}$, so that $k_3 = 0$, a contradiction. Thus $\overline{\varphi_+}$ and $\overline{\psi_+}$ are of bounded type, so that Φ^* is of bounded type. Since T_Φ is hyponormal, it follows from (8) that Φ is also of bounded type, giving (33). Thus we can write

$$\varphi_- := \theta_0 \overline{a} \quad \text{and} \quad \psi_- := \theta_1 \overline{b} \quad (a \in \mathcal{H}_{z\theta_0},\, b \in \mathcal{H}_{z\theta_1}),$$

where θ_0 and θ_1 are inner, a and θ_0 are coprime and b and θ_1 are coprime. On the other hand, by (34), we have

$$\begin{cases} \overline{b}_\alpha - k_2\overline{\varphi_+} \in H^2, & \overline{\theta}_1 b - k_4 \overline{\varphi_+} \in H^2 \\ \overline{b}_\beta - k_3\overline{\psi_+} \in H^2, & \overline{\theta}_0 a - k_1 \overline{\psi_+} \in H^2, \end{cases} \tag{35}$$

which implies that the following Toeplitz operators are all hyponormal (by Cowen's Theorem):

$$T_{\overline{b}_\alpha + \varphi_+}, \quad T_{\overline{\theta}_1 b + \varphi_+}, \quad T_{\overline{b}_\beta + \psi_+}, \quad T_{\overline{\theta}_0 a + \psi_+}. \tag{36}$$

Then by the scalar-valued version of Lemma 3.1, we can write

$$\varphi_+ = \theta_1\theta_3 \overline{d} \quad \text{and} \quad \psi_+ = \theta_0\theta_2\overline{c} \quad (d \in \mathcal{H}_{z\theta_1\theta_3},\, c \in \mathcal{H}_{z\theta_0\theta_2}), \tag{37}$$

where θ_2 and θ_3 are inner, d and $\theta_1\theta_3$ are coprime, and c and $\theta_0\theta_2$ are coprime. In particular, $d(\alpha) \neq 0$ and $c(\beta) \neq 0$. We now claim that

$$\alpha = \beta. \tag{38}$$

Assume to the contrary that $\alpha \neq \beta$. Since Φ is normal, i.e., $\Phi\Phi^* = \Phi^*\Phi$, we have

$$\begin{bmatrix} \overline{b}_\alpha & \varphi \\ \psi & \overline{b}_\beta \end{bmatrix} \begin{bmatrix} b_\alpha & \overline{\psi} \\ \overline{\varphi} & b_\beta \end{bmatrix} = \begin{bmatrix} b_\alpha & \overline{\psi} \\ \overline{\varphi} & b_\beta \end{bmatrix} \begin{bmatrix} \overline{b}_\alpha & \varphi \\ \psi & \overline{b}_\beta \end{bmatrix},$$

which gives

$$\overline{b}_\alpha \overline{\psi} + \varphi b_\beta = b_\alpha \varphi + \overline{\psi}\overline{b}_\beta, \quad \text{i.e.,} \quad (b_\alpha - b_\beta)(\psi + \overline{b}_\alpha \overline{b}_\beta \overline{\varphi}) = 0,$$

which implies that $\psi = -\overline{b}_\alpha \overline{b}_\beta \overline{\varphi}$ since $\alpha \neq \beta$. We put

$$\varphi_-' := P_{\mathcal{H}(b_\alpha b_\beta)}(\varphi_-) \quad \text{and} \quad \varphi_-'' := P_{b_\alpha b_\beta H^2}(\varphi_-).$$

We then have

$$\psi_+ = -\overline{b}_\alpha \overline{b}_\beta \varphi_-'' \quad \text{and} \quad \psi_- = -b_\alpha b_\beta(\varphi_+ + \overline{\varphi_-'}). \tag{39}$$

It thus follows from (39) that

$$\theta_1 \overline{b} = \psi_- = -b_\alpha b_\beta(\varphi_+ + \overline{\varphi_-'}), \quad \text{so that} \quad \overline{b} = -b_\alpha b_\beta(\theta_3 \overline{d} + \overline{\theta_1 \varphi_-'}) \in \overline{H^2} \tag{40}$$

which gives

$$\theta_3\overline{d}+\overline{\theta_1\varphi'_-}\in\overline{H^2},\quad \text{and hence, } \ d\in\theta_3H^2,$$

which implies that θ_3 is a constant because θ_3 and d are coprime. We therefore have $\varphi_+=\theta_1\overline{d}$. It thus follows from (36) together with again Lemma 3.1 that

$$\theta_1=b_\alpha\theta'_1\quad \text{(some inner function } \theta'_1).$$

But since by (40),

$$\overline{b}=-b_\alpha b_\beta\overline{(d+\theta_1\varphi'_-)}\in\overline{H^2},$$

so that

$$d+\theta_1\varphi'_-\in b_\alpha b_\beta H^2,$$

which implies that $d(\alpha)=0$, a contradiction because θ_1 and d are coprime. This proves (38).

We now write

$$\Phi\equiv\begin{bmatrix}\overline{b}_\alpha & \varphi\\ \psi & \overline{b}_\alpha\end{bmatrix}\equiv\Phi_-^*+\Phi_+=\begin{bmatrix}b_\alpha & \psi_-\\ \varphi_- & b_\alpha\end{bmatrix}^*+\begin{bmatrix}0 & \varphi_+\\ \psi_+ & 0\end{bmatrix},$$

where

$$\varphi_-:=\theta_0\overline{a}\quad\text{and}\quad\psi_-:=\theta_1\overline{b}\qquad(a\in\mathcal{H}_{\theta_0},\ b\in\mathcal{H}_{\theta_1}).$$

Moreover, we have

$$\begin{cases}\overline{b}_\alpha-k_2\overline{\varphi_+}\in H^2, & \overline{\theta}_1b-k_4\overline{\varphi_+}\in H^2\\ \overline{b}_\alpha-k_3\overline{\psi_+}\in H^2, & \overline{\theta}_0a-k_1\overline{\psi_+}\in H^2\end{cases}\tag{41}$$

and the following Toeplitz operators are all hyponormal:

$$T_{\overline{b_\alpha+\varphi_+}},\ T_{\overline{\theta_1b+\varphi_+}},\ T_{\overline{b_\alpha+\psi_+}},\ T_{\overline{\theta_0a+\psi_+}}.\tag{42}$$

Note that $\varphi_+\psi_+$ is not identically zero, so that $\det\Phi_+$ is not. Put

$$\theta_0=b_\alpha^m\theta'_0\quad\text{and}\quad\theta_1=b_\alpha^n\theta'_1\qquad(m,n\ge0;\ \theta'_0(\alpha)\ne0,\ \theta'_1(\alpha)\ne0).$$

We now claim that

$$m=n=0\quad\text{or}\quad m=n=1.\tag{43}$$

We split the proof of (43) into three cases.

Case 1 ($m\ne0$ and $n=0$): In this case, we have $a(\alpha)\ne0$ because $\theta_0(\alpha)=0$ and θ_0 and a are coprime. We first claim that

$$m=1.\tag{44}$$

To show this we assume to the contrary that $m\ge2$. Write

$$\gamma:=-\frac{a(\alpha)}{\theta_1(\alpha)}\quad\text{and}\quad\nu:=\frac{1}{\sqrt{|\gamma|^2+1}}.$$

To get the left coprime factorization of Φ_-, applying Lemma 4.1(b) for $\widetilde{\Phi_-}$ gives

$$\widetilde{\Phi_-}=\begin{bmatrix}\widetilde{b}_\alpha & \widetilde{\theta}_0\overline{\widetilde{a}}\\ \widetilde{\theta}_1\overline{\widetilde{b}} & \widetilde{b}_\alpha\end{bmatrix}=\widetilde{\Omega}_2\widetilde{B}^*\quad\text{(right coprime factorization)},$$

where

$$\Omega_2:=\nu\begin{bmatrix}b_\alpha^{m-1}\theta'_0 & \gamma\theta_1\\ -\overline{\gamma}\theta_0 & b_\alpha\theta_1\end{bmatrix}\quad\text{and}\quad B\in H^2_{M_n},$$

which gives

$$\Phi_-=\begin{bmatrix}b_\alpha & \theta_1\overline{b}\\ \theta_0\overline{a} & b_\alpha\end{bmatrix}=B^*\Omega_2\quad\text{(left coprime factorization)}.$$

On the other hand, by a scalar-valued version of Lemma 3.1 and (42), we can see that

$$\varphi_+=b_\alpha\theta_1\theta_3\overline{d}\ \text{ and }\ \psi_+=\theta_0\theta_2\overline{c}\ \text{ for some inner functions } \theta_2,\theta_3,$$

where $d\in\mathcal{H}_{zb_\alpha\theta_1\theta_3}$ and $c\in\mathcal{H}_{z\theta_0\theta_2}$. Thus in particular, $c(\alpha)\ne0$ and $d(\alpha)\ne0$. We first observe that

$$k_3(\alpha)=0\quad\text{and}\quad k_4(\alpha)=0:\tag{45}$$

indeed, in (41),

$$\begin{aligned}\overline{b}_\alpha - k_3\overline{\psi_+} \in H^2 &\Longrightarrow \overline{b}_\alpha - k_3\overline{\theta_0\theta_2}c \in H^2\\ &\Longrightarrow b_\alpha^{m-1}\theta_0'\theta_2 - k_3 c \in b_\alpha^m\theta_0'\theta_2 H^2\\ &\Longrightarrow k_3(\alpha)=0 \quad (\text{since } m\geq 2)\end{aligned}$$

and

$$\begin{aligned}\overline{\theta}_1 b - k_4\overline{\varphi_+} \in H^2 &\Longrightarrow \overline{\theta}_1 b - k_4\overline{b}_\alpha\overline{\theta_1\theta_3}d \in H^2\\ &\Longrightarrow b_\alpha\theta_3 b - k_4 d \in b_\alpha\theta_1\theta_3 H^2\\ &\Longrightarrow k_4(\alpha)=0\,,\end{aligned}$$

which proves (45). Write

$$\theta_2 = b_\alpha^q\theta_2' \quad \text{and} \quad \theta_3 = b_\alpha^p\theta_3' \quad (\theta_2'(\alpha)\neq 0,\ \theta_3'(\alpha)\neq 0).$$

Then we can write

$$\Phi_+ = \begin{bmatrix} 0 & b_\alpha\theta_1\theta_3\overline{d}\\ \theta_0\theta_2\overline{c} & 0\end{bmatrix} = \begin{bmatrix} 0 & b_\alpha^{p+1}\theta_1\theta_3'\overline{d}\\ b_\alpha^{m+q}\theta_0'\theta_2'\overline{c} & 0\end{bmatrix}.$$

We suppose that $p+1 < m+q$ and write $r := (m+q)-(p+1) > 0$. Then

$$\Phi_+ = (b_\alpha^{m+q}\theta_1\theta_3'\theta_0'\theta_2')I_2 \begin{bmatrix} 0 & \theta_1\theta_3'c\\ b_\alpha^r\theta_0'\theta_2'd & 0\end{bmatrix}^* \equiv (\theta I_2)A^*,$$

where $\theta := b_\alpha^{m+q}\theta_1\theta_3'\theta_0'\theta_2'$. Observe that

$$A\Omega_2^* = \nu\begin{bmatrix} 0 & \theta_1\theta_3'c\\ b_\alpha^r\theta_0'\theta_2'd & 0\end{bmatrix}\begin{bmatrix} b_\alpha^{m-1}\theta_0' & \gamma\theta_1\\ -\overline{\gamma}\theta_0 & b_\alpha\theta_1\end{bmatrix}^* = \nu\begin{bmatrix}\overline{\gamma}\theta_3'c & \overline{b}_\alpha\theta_3'c\\ b_\alpha^{r-m+1}\theta_2'd & -\gamma b_\alpha^{r-m}\theta_2'd\end{bmatrix}.$$

If $r\geq m-1$, then we have

$$H_{A\Omega_2^*}\begin{bmatrix}0\\1\end{bmatrix} = \nu\begin{bmatrix} H_{\overline{b}_\alpha}(\theta_3'c)\\ -\gamma H_{b_\alpha^{r-m}}(\theta_2'd)\end{bmatrix}.$$

Put

$$\delta := \frac{\sqrt{1-|\alpha|^2}}{1-\overline{\alpha}z} \quad \text{and} \quad \delta_1 := \widetilde{\delta} = \frac{\sqrt{1-|\alpha|^2}}{1-\alpha z}$$

and observe that $\operatorname{ran} H_{\overline{b}_\alpha} = \mathcal{H}_{b_{\overline{\alpha}}} = \bigvee\{\delta_1\}$. Since $(\theta_3'c)(\alpha)\neq 0$ and $(\theta_2'd)(\alpha)\neq 0$, it follows from (21) that

$$\begin{bmatrix}\delta_1\\ \beta\delta_1\end{bmatrix} \in \text{cl ran}\, H_{A\Omega_2^*} \subseteq \ker(I - T_{\widetilde{K}}T_{\widetilde{K}}^*), \tag{46}$$

where $\beta\in\mathbb{C}$ is possibly zero (when $r\geq m$). We observe that if $k\in H^2$, then since $\frac{1}{1-\alpha z}$ is the reproducing kernel for $\overline{\alpha}$, we can get

$$T_{k(\overline{z})}\delta_1 = k(\alpha)\delta_1\,: \tag{47}$$

indeed, if $k\in H^2$ and $n\geq 0$, then

$$\langle k(\overline{z})\delta_1,\ z^n\rangle = \langle \delta_1,\ \overline{k(\overline{z})}z^n\rangle = \overline{\langle \widetilde{k}z^n,\ \delta_1\rangle} = \sqrt{1-|\alpha|^2}\,\overline{\widetilde{k}(\overline{\alpha})\overline{\alpha}^n} = \sqrt{1-|\alpha|^2}\,k(\alpha)\alpha^n\,,$$

so that

$$T_{k(\overline{z})}\delta_1 = P(k(\overline{z})\delta_1) = \sqrt{1-|\alpha|^2}\,k(\alpha)\sum_{n=0}^\infty \alpha^n z^n = k(\alpha)\frac{\sqrt{1-|\alpha|^2}}{1-\alpha z} = k(\alpha)\delta_1\,,$$

which proves (47). It thus follows from (45), (46) and (47) that

$$
\begin{aligned}
\begin{bmatrix}\delta_1\\ \beta\delta_1\end{bmatrix} = T_{\widetilde{K}}T^*_{\widetilde{K}}\begin{bmatrix}\delta_1\\ \beta\delta_1\end{bmatrix} &= \begin{bmatrix}T_{\widetilde{k_1}} & T_{\widetilde{k_3}}\\ T_{\widetilde{k_2}} & T_{\widetilde{k_4}}\end{bmatrix}\begin{bmatrix}T_{\overline{\widetilde{k_1}}} & T_{\overline{\widetilde{k_2}}}\\ T_{\overline{\widetilde{k_3}}} & T_{\overline{\widetilde{k_4}}}\end{bmatrix}\begin{bmatrix}\delta_1\\ \beta\delta_1\end{bmatrix}\\
&= \begin{bmatrix}T_{\widetilde{k_1}} & T_{\widetilde{k_3}}\\ T_{\widetilde{k_2}} & T_{\widetilde{k_4}}\end{bmatrix}\begin{bmatrix}T_{k_1(\overline{z})}\delta_1+\beta T_{k_2(\overline{z})}\delta_1\\ T_{k_3(\overline{z})}\delta_1+\beta T_{k_4(\overline{z})}\delta_1\end{bmatrix}\\
&= \begin{bmatrix}T_{\widetilde{k_1}} & T_{\widetilde{k_3}}\\ T_{\widetilde{k_2}} & T_{\widetilde{k_4}}\end{bmatrix}\begin{bmatrix}(k_1(\alpha)+\beta k_2(\alpha))\delta_1\\ 0\end{bmatrix}\\
&= \begin{bmatrix}\widetilde{k_1}(k_1(\alpha)+\beta k_2(\alpha))\delta_1\\ \widetilde{k_2}(k_1(\alpha)+\beta k_2(\alpha))\delta_1\end{bmatrix},
\end{aligned}
$$

which implies that k_1 and k_2 are nonzero constants. Thus by (41),

$$\overline{b}_\alpha - k_2\overline{\varphi_+}\in H^2 \Longrightarrow b_\alpha\overline{\varphi}_+\in H^2 \Longrightarrow \overline{\theta_1\theta_3}d\in H^2\,,$$

which implies that $\theta_1\theta_3$ is a constant. Without loss of generality we may assume $\theta_1\theta_3=1$ and $\psi_-=0$. Similarly, from (41), $\overline{\theta_0}a-k_1\overline{\psi_+}\in H^2$, i.e., $\overline{\theta_0}a-k_1\overline{\theta_0\theta_2}c\in H^2$ implies $\theta_2=1$. But since by (32), $|\varphi|=|\psi|$, we have

$$|b_\alpha\overline{d}+\overline{\theta_0}a| = |\varphi_+ + \overline{\varphi_-}| = |\psi_+| = |\theta_0\overline{c}| \quad (\text{where } a\in\mathcal{H}_{\theta_0},\ c\in\mathcal{H}_{z\theta_0}\ ),$$

which implies

$$b_\alpha\theta_0(b_\alpha\overline{d}+\overline{\theta_0}a)(\overline{b}_\alpha d+\theta_0\overline{a}) = b_\alpha\theta_0 c\overline{c}\,,$$

so that

$$ad = b_\alpha\Big((\theta_0\overline{c})c-(\theta_0\overline{d})d-(\theta_0\overline{a})(\theta_0\overline{d})b_\alpha-(\theta_0\overline{a})a\Big)\,. \tag{48}$$

But since $a,c\in\mathcal{H}_{z\theta_0}$, and $d\in\mathcal{H}_{zb_\alpha}$, it follows that $\theta_0\overline{a},\theta_0\overline{c}$ and $\theta_0\overline{d}$ are in H^2. Thus (48) implies that $(ad)(\alpha)=0$, a contradiction. Therefore this case cannot occur.

If instead $r<m-1$ then the same argument as before leads to a contradiction. Therefore this case cannot occur. Moreover, by the same argument as in the case $p+1<m+q$, the case $p+1\geq m+q$ cannot also occur. This proves (44).

Now by Lemma 4.1(b), we can write

$$\Phi_- = \begin{bmatrix}\theta_0' & a\\ b_\alpha b & \theta_1\end{bmatrix}^*\begin{bmatrix}\theta_0 & 0\\ 0 & b_\alpha\theta_1\end{bmatrix} \equiv B^*\Omega_2 \ \ \text{(left coprime factorization).}$$

Observe

$$\Omega_2 \equiv \begin{bmatrix}\theta_0 & 0\\ 0 & b_\alpha\theta_1\end{bmatrix} = \begin{bmatrix}b_\alpha & 0\\ 0 & b_\alpha\end{bmatrix}\begin{bmatrix}\theta_0' & 0\\ 0 & \theta_1\end{bmatrix}.$$

Since $\theta_0'(\alpha)\neq 0$ and $\theta_1(\alpha)\neq 0$, it follows from Lemma 3.6 that $\begin{bmatrix}b_\alpha & 0\\ 0 & b_\alpha\end{bmatrix}$ and $\begin{bmatrix}\theta_0' & 0\\ 0 & \theta_1\end{bmatrix}$ are coprime, so that by Corollary 3.9, T_Φ should be normal. Since $\det\Phi_+$ is not identically zero, it follows form Lemma 1.2 that $\Phi_+-\Phi_-U\in M_n$ for some constant unitary matrix $U\equiv\left[\begin{smallmatrix}c_1 & c_2\\ c_3 & c_4\end{smallmatrix}\right]$. We observe

$$
\begin{aligned}
\Phi_+-\Phi_-U\in M_n \Longleftrightarrow &\begin{bmatrix}0 & b_\alpha\theta_1\theta_3\overline{d}\\ \theta_0\theta_2\overline{c} & 0\end{bmatrix} - \begin{bmatrix}b_\alpha & \theta_1\overline{b}\\ \theta_0\overline{a} & b_\alpha\end{bmatrix}\begin{bmatrix}c_1 & c_2\\ c_3 & c_4\end{bmatrix} = \begin{bmatrix}\xi_1 & \xi_2\\ \xi_3 & \xi_4\end{bmatrix} \quad (\xi_i\in\mathbb{C})\\
\Longrightarrow &\begin{cases}c_1b_\alpha+c_3\theta_1\overline{b}=-\xi_1\\ c_4b_\alpha+c_2\theta_0\overline{a}=-\xi_4\end{cases}\\
\Longrightarrow &\ c_1=0,\ c_4\neq 0\\
\Longrightarrow &\ U=\begin{bmatrix}0 & c_2\\ c_3 & c_4\end{bmatrix} \quad (c_4\neq 0),
\end{aligned}
$$

which contradicts the fact that U is unitary. Thus this case cannot occur.

Case 2 ($m=0$ and $n\neq 0$): This case is symmetrical to Case 1. The proof is identical to that of Case 1. Therefore this case cannot occur either.

Case 3 ($m \neq 0$, $n \neq 0$ and $m \geq 2$ or $n \geq 2$): In this case, $a(\alpha) \neq 0$ and $b(\alpha) \neq 0$ since θ_0 and a are coprime and θ_1 and b are coprime. By Lemma 4.1(d), we can write

$$\Phi_- = \begin{bmatrix} b_\alpha^{m-1}\theta_0' & a \\ b & b_\alpha^{n-1}\theta_1' \end{bmatrix}^* \begin{bmatrix} \theta_0 & 0 \\ 0 & \theta_1 \end{bmatrix} \equiv B^*\Omega_2 \ \ \text{(left coprime factorization)}.$$

We first suppose $m \geq 2$. By a scalar-valued version of Lemma 3.1 and (42), we can see that

$$\varphi_+ = \theta_1\theta_3\overline{d} \ \ \text{and} \ \ \psi_+ = \theta_0\theta_2\overline{c} \ \ \text{for some inner functions } \theta_2, \theta_3,$$

where $d \in \mathcal{H}_{z\theta_1\theta_3}$ and $c \in \mathcal{H}_{z\theta_0\theta_2}$. Note that $c(\alpha) \neq 0$ and $d(\alpha) \neq 0$. We first observe

$$k_3(\alpha) = 0: \tag{49}$$

indeed, in (41), $\overline{b}_\alpha - k_3\overline{\psi_+} \in H^2$ implies $b_\alpha^{m-1}\theta_0'\theta_2 - k_3 c \in b_\alpha^m\theta_0'\theta_2 H^2$, giving $k_3(\alpha) = 0$. Write

$$\theta_2 = b_\alpha^q\theta_2' \quad \text{and} \quad \theta_3 = b_\alpha^p\theta_3' \quad (\theta_2'(\alpha) \neq 0,\, \theta_3'(\alpha) \neq 0).$$

Then we can write

$$\Phi_+ = \begin{bmatrix} 0 & \theta_1\theta_3\overline{d} \\ \theta_0\theta_2\overline{c} & 0 \end{bmatrix} = \begin{bmatrix} 0 & b_\alpha^{n+p}\,\theta_1'\theta_3'\overline{d} \\ b_\alpha^{m+q}\,\theta_0'\theta_2'\overline{c} & 0 \end{bmatrix}.$$

If $n + p \leq m + q$, write $r := (m+q) - (n+p) \geq 0$. Then

$$\Phi_+ = (b_\alpha^{m+q}\theta_1'\theta_3'\theta_0'\theta_2')I_2 \begin{bmatrix} 0 & \theta_1'\theta_3'c \\ b_\alpha^r\theta_0'\theta_2'd & 0 \end{bmatrix}^* \equiv (\theta I_2)A^*,$$

where $\theta := b_\alpha^{m+q}\theta_1'\theta_3'\theta_0'\theta_2'$. Observe that

$$A\Omega_2^* = \begin{bmatrix} 0 & \theta_1'\theta_3'c \\ b_\alpha^r\theta_0'\theta_2'd & 0 \end{bmatrix} \begin{bmatrix} \theta_0 & 0 \\ 0 & \theta_1 \end{bmatrix}^* = \begin{bmatrix} 0 & \overline{b}_\alpha^n\theta_3'c \\ b_\alpha^{r-m}\theta_2'd & 0 \end{bmatrix},$$

so that

$$H_{A\Omega_2^*}\begin{bmatrix} 0 \\ b_\alpha^{n-1} \end{bmatrix} = \begin{bmatrix} H_{\overline{b}_\alpha}(\theta_3'c) \\ 0 \end{bmatrix}.$$

Since $(\theta_3'c)(\alpha) \neq 0$, it follows from (21) and the same argument as (46) that

$$\begin{bmatrix} \delta_1 \\ 0 \end{bmatrix} \in \text{cl ran}\, H_{A\Omega_2^*} \subseteq \ker(I - T_{\widetilde{K}}T_{\widetilde{K}}^*).$$

Since by (49), $k_3(\alpha) = 0$, it follows from (47) that

$$\begin{aligned} \begin{bmatrix} \delta_1 \\ 0 \end{bmatrix} = T_{\widetilde{K}}T_{\widetilde{K}}^* \begin{bmatrix} \delta_1 \\ 0 \end{bmatrix} &= \begin{bmatrix} T_{\widetilde{k_1}} & T_{\widetilde{k_3}} \\ T_{\widetilde{k_2}} & T_{\widetilde{k_4}} \end{bmatrix} \begin{bmatrix} T_{\overline{\widetilde{k}}_1} & T_{\overline{\widetilde{k}}_2} \\ T_{\overline{\widetilde{k}}_3} & T_{\overline{\widetilde{k}}_4} \end{bmatrix} \begin{bmatrix} \delta_1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} T_{\widetilde{k_1}} & T_{\widetilde{k_3}} \\ T_{\widetilde{k_2}} & T_{\widetilde{k_4}} \end{bmatrix} \begin{bmatrix} k_1(\alpha)\delta_1 \\ k_3(\alpha)\delta_1 \end{bmatrix} \\ &= \begin{bmatrix} k_1(\alpha)\widetilde{k}_1\delta_1 \\ k_1(\alpha)\widetilde{k}_2\delta_1 \end{bmatrix}, \end{aligned}$$

which implies that $k_2 = 0$. This leads to a contradiction with (41). If $n+p > m+q$, then a similar argument leads to a contradiction. Also if instead $n \geq 2$, then the argument is symmetrical with the case $m \geq 2$. Thus this case cannot occur.

Consequently, all three cases cannot occur. This proves (43).

Now it suffices to consider the case $m = n = 0$ and the case $m = n = 1$.

Case A ($m = n = 0$) In this case, by Lemma 4.1(c) we can write

$$\Phi_- = \begin{bmatrix} \theta_0 & b_\alpha a \\ b_\alpha b & \theta_1 \end{bmatrix}^* \begin{bmatrix} b_\alpha\theta_0 & 0 \\ 0 & b_\alpha\theta_1 \end{bmatrix} \equiv B^*\Omega_2 \ \ \text{(left coprime factorization)}.$$

Observe that

$$\Omega_2 \equiv \begin{bmatrix} b_\alpha\theta_0 & 0 \\ 0 & b_\alpha\theta_1 \end{bmatrix} = \begin{bmatrix} b_\alpha & 0 \\ 0 & b_\alpha \end{bmatrix} \begin{bmatrix} \theta_0 & 0 \\ 0 & \theta_1 \end{bmatrix}.$$

Since $\theta_0(\alpha)\neq 0$ and $\theta_1(\alpha)\neq 0$, it follows from Lemma 3.6 that $\begin{bmatrix} b_\alpha & 0\\ 0 & b_\alpha\end{bmatrix}$ and $\begin{bmatrix} \theta_0 & 0\\ 0 & \theta_1\end{bmatrix}$ are coprime, so that by Corollary 3.9, T_Φ should be normal. Since $\det \Phi_+$ is not identically zero, it follows form Lemma 1.2 that $\Phi_+ - \Phi_- U \in M_n$ for some constant unitary matrix $U \equiv \left[\begin{smallmatrix} c_1 & c_2\\ c_3 & c_4\end{smallmatrix}\right]$. We observe

$$\begin{aligned}
\Phi_+ - \Phi_- U \in M_n &\Longleftrightarrow \begin{bmatrix} 0 & \theta_1\theta_3\overline{d}\\ \theta_0\theta_2\overline{c} & 0\end{bmatrix} - \begin{bmatrix} b_\alpha & \theta_1\overline{b}\\ \theta_0\overline{a} & b_\alpha\end{bmatrix}\begin{bmatrix} c_1 & c_2\\ c_3 & c_4\end{bmatrix} = \begin{bmatrix} \xi_1 & \xi_2\\ \xi_3 & \xi_4\end{bmatrix} \quad (\xi_i \in \mathbb{C})\\
&\Longleftrightarrow \begin{cases} c_1 b_\alpha + c_3\theta_1\overline{b} = -\xi_1\\ c_4 b_\alpha + c_2\theta_0\overline{a} = -\xi_4\\ \theta_1\theta_3\overline{d} = c_2 b_\alpha + c_4\theta_1\overline{b} + \xi_2\\ \theta_0\theta_2\overline{c} = c_3 b_\alpha + c_1\theta_0\overline{a} + \xi_3\end{cases}\\
&\Longrightarrow \begin{cases} c_1 = 0,\ \theta_1\overline{b} = \text{a constant}\\ c_4 = 0,\ \theta_0\overline{a} = \text{a constant}\\ \varphi_+ = \theta_1\theta_3\overline{d} = c_2 b_\alpha + \text{a constant}\\ \psi_+ = \theta_0\theta_2\overline{c} = c_3 b_\alpha + \text{a constant}\,.\end{cases}
\end{aligned} \tag{50}$$

Since U is unitary we have $c_2 = e^{i\omega_1}$ and $c_3 = e^{i\omega_2}$ $(\omega_1,\omega_2\in[0,2\pi))$. Thus we have

$$\varphi = e^{i\omega_1} b_\alpha + \beta_1 \quad \text{and} \quad \psi = e^{i\omega_2} b_\alpha + \beta_2 \quad (\beta_1,\beta_2\in\mathbb{C}).$$

But since $|\varphi| = |\psi|$, it follows that

$$\varphi = e^{i\omega} b_\alpha + \zeta \quad \text{and} \quad \psi = e^{i\delta}\varphi \quad (\omega,\delta\in[0,2\pi),\ \zeta\in\mathbb{C})).$$

Case B ($m = n = 1$): We split the proof into two subcases.

Case B-1 ($m = n = 1$; $(ab)(\alpha)\neq(\theta_0'\theta_1')(\alpha)$): In this case, by Lemma 4.1(d), we can write

$$\Phi_- = \begin{bmatrix} \theta_0' & a\\ b & \theta_1'\end{bmatrix}^* \begin{bmatrix} b_\alpha\theta_0' & 0\\ 0 & b_\alpha\theta_1'\end{bmatrix} \equiv B^*\Omega_2 \quad \text{(left coprime factorization).}$$

Observe

$$\Omega_2 \equiv \begin{bmatrix} b_\alpha\theta_0' & 0\\ 0 & b_\alpha\theta_1'\end{bmatrix} = \begin{bmatrix} b_\alpha & 0\\ 0 & b_\alpha\end{bmatrix}\begin{bmatrix} \theta_0' & 0\\ 0 & \theta_1'\end{bmatrix}.$$

Since $\theta_0'(\alpha)\neq 0$ and $\theta_1'(\alpha)\neq 0$, it follows from Lemma 3.6 that $\begin{bmatrix} b_\alpha & 0\\ 0 & b_\alpha\end{bmatrix}$ and $\begin{bmatrix} \theta_0' & 0\\ 0 & \theta_1'\end{bmatrix}$ are coprime, so that by Corollary 3.9, T_Φ should be normal. By the same argument as in (50), we can see that

$$\theta_2 = \theta_3 = 1 \quad \text{and} \quad \theta_0' = \theta_1' = 1$$

and we can write

$$\Phi_+ = \begin{bmatrix} 0 & \varphi_+\\ \psi_+ & 0\end{bmatrix} \quad \text{and} \quad \Phi_-^* = \begin{bmatrix} \overline{b}_\alpha & a\,\overline{b}_\alpha\\ b\,\overline{b}_\alpha & \overline{b}_\alpha\end{bmatrix} \qquad (a,b\in\mathbb{C};\ a\neq 0,\ b\neq 0).$$

Since T_Φ is normal we have

$$\begin{bmatrix} H^*_{\overline{\varphi_+}}H_{\overline{\varphi_+}} & 0\\ 0 & H^*_{\overline{\psi_+}}H_{\overline{\psi_+}}\end{bmatrix} = \begin{bmatrix} (1+|b|^2)H^*_{\overline{b}_\alpha}H_{\overline{b}_\alpha} & (a+\overline{b})H^*_{\overline{b}_\alpha}H_{\overline{b}_\alpha}\\ (\overline{a}+b)H^*_{\overline{b}_\alpha}H_{\overline{b}_\alpha} & (1+|a|^2)H^*_{\overline{b}_\alpha}H_{\overline{b}_\alpha}\end{bmatrix},$$

which implies that

$$\begin{cases} b = -\overline{a}\\ H^*_{\overline{\varphi_+}}H_{\overline{\varphi_+}} = (1+|b|^2)H^*_{\overline{b}_\alpha}H_{\overline{b}_\alpha}\\ H^*_{\overline{\psi_+}}H_{\overline{\psi_+}} = (1+|a|^2)H^*_{\overline{b}_\alpha}H_{\overline{b}_\alpha}\,.\end{cases} \tag{51}$$

Since $ab\neq(\theta_0'\theta_1')(\alpha)$, we have $1\neq|ab| = |a|^2$, i.e., $|a|\neq 1$. We thus have

$$\varphi_+ = e^{i\theta_1}\sqrt{1+|a|^2}\, b_\alpha + \beta_1 \quad \text{and} \quad \psi_+ = e^{i\theta_2}\sqrt{1+|a|^2}\, b_\alpha + \beta_2,$$

$(\beta_1,\beta_2\in\mathbb{C};\ \theta_1,\theta_2\in[0,2\pi))$ which implies that

$$\varphi = a\,\overline{b}_\alpha + e^{i\theta_1}\sqrt{1+|a|^2}\, b_\alpha + \beta_1 \quad \text{and} \quad \psi = -\overline{a}\,\overline{b}_\alpha + e^{i\theta_2}\sqrt{1+|a|^2}\, b_\alpha + \beta_2.$$

Since $|\varphi| = |\psi|$, a straightforward calculation shows that

$$\varphi = \mu\,\overline{b}_\alpha + e^{i\theta}\sqrt{1+|\mu|^2}\,b_\alpha + \zeta \quad \text{and} \quad \psi = e^{i\,(\pi - 2\arg\mu)}\varphi, \tag{52}$$

where $\mu \neq 0,\ |\mu| \neq 1,\ \zeta \in \mathbb{C}$, and $\theta \in [0, 2\pi)$.

Case B-2 ($m = n = 1$; $(ab)(\alpha) = (\theta_0'\theta_1')(\alpha)$): In this case, $\theta_i'(\alpha) \neq 0$ for each $i = 0, 1$. By a scalar-valued version of Lemma 3.1 and (36), we can see that

$$\varphi_+ = \theta_1\theta_3\overline{d} \ \text{ and } \ \psi_+ = \theta_0\theta_2\overline{c} \ \text{ for some inner functions } \theta_2, \theta_3,$$

where $d \in \mathcal{H}_{z\theta_1\theta_3}$ and $c \in \mathcal{H}_{z\theta_0\theta_2}$. Thus in particular, $c(\alpha) \neq 0$ and $d(\alpha) \neq 0$. Let

$$\theta_2 = b_\alpha^q\theta_2' \quad \text{and} \quad \theta_3 = b_\alpha^p\theta_3' \quad (\theta_2'(\alpha) \neq 0,\ \theta_3'(\alpha) \neq 0).$$

To get the left coprime factorization of Φ_-, applying Lemma 4.1(d) for $\widetilde{\Phi}_-$ gives

$$\widetilde{\Phi}_- = \begin{bmatrix} \widetilde{b}_\alpha & \widetilde{\theta}_0\widetilde{\overline{a}} \\ \widetilde{\theta}_1\widetilde{\overline{b}} & \widetilde{b}_\alpha \end{bmatrix} = \widetilde{\Omega}_2\widetilde{B}^* \quad \text{(right coprime factorization)},$$

where

$$\Omega_2 := \nu\begin{bmatrix} \theta_0 & -\overline{\gamma}\theta_1 \\ \gamma\theta_0' & \theta_1' \end{bmatrix} \quad (\gamma = -\frac{b(\alpha)}{\theta_0'(\alpha)} = -\frac{\theta_1'(\alpha)}{a(\alpha)})$$

Then we get

$$\Phi_- = \begin{bmatrix} b_\alpha & \theta_1\overline{b} \\ \theta_0\overline{a} & b_\alpha \end{bmatrix} = B^*\Omega_2 \quad \text{(left coprime factorization)}\,. \tag{53}$$

We now claim that

$$p = q. \tag{54}$$

We first assume that $p < q$. Then $\theta_2(\alpha) = 0$. Thus by (41), we have $k_1(\alpha) = k_3(\alpha) = 0$. Write $s := q - p \geq 1$. In this case we can write

$$\Phi_+ = \begin{bmatrix} 0 & \theta_1\theta_3\overline{d} \\ \theta_0\theta_2\overline{c} & 0 \end{bmatrix} = (b_\alpha^{q+1}\theta_1'\theta_3'\theta_0'\theta_2')I_2\begin{bmatrix} 0 & \theta_1'\theta_3'c \\ b_\alpha^s\theta_0'\theta_2'd & 0 \end{bmatrix}^* \equiv (\theta I_2)A^*\,,$$

where $\theta := b_\alpha^{q+1}\theta_1'\theta_3'\theta_0'\theta_2'$. Observe that

$$A\Omega_2^* = \nu\begin{bmatrix} 0 & \theta_1'\theta_3'c \\ b_\alpha^s\theta_0'\theta_2'd & 0 \end{bmatrix}\begin{bmatrix} \theta_0 & -\overline{\gamma}\theta_1 \\ \gamma\theta_0' & \theta_1' \end{bmatrix}^* = \nu\begin{bmatrix} -\gamma\overline{b}_\alpha\theta_3'c & \theta_3'c \\ b_\alpha^{s-1}\theta_2'd & \overline{\gamma}b_\alpha^s\theta_2'd \end{bmatrix}\,. \tag{55}$$

Since $s \geq 1$, we have

$$H_{A\Omega_2^*}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \nu\begin{bmatrix} -\gamma H_{\overline{b}_\alpha}(\theta_3'c) \\ 0 \end{bmatrix}.$$

Since $(\theta_3'c)(\alpha) \neq 0$, it follows from (21) that

$$\begin{bmatrix} \delta_1 \\ 0 \end{bmatrix} \in \text{cl ran}\, H_{A\Omega_2^*} \subseteq \ker\,(I - T_{\widetilde{K}}T_{\widetilde{K}}^*).$$

Thus we have

$$\begin{aligned} \begin{bmatrix} \delta_1 \\ 0 \end{bmatrix} = T_{\widetilde{K}}T_{\widetilde{K}}^*\begin{bmatrix} \delta_1 \\ 0 \end{bmatrix} &= \begin{bmatrix} T_{\widetilde{k_1}} & T_{\widetilde{k_3}} \\ T_{\widetilde{k_2}} & T_{\widetilde{k_4}} \end{bmatrix}\begin{bmatrix} T_{k_1(\overline{z})}\delta_1 \\ T_{k_3(\overline{z})}\delta_1 \end{bmatrix} \\ &= \begin{bmatrix} T_{\widetilde{k_1}} & T_{\widetilde{k_3}} \\ T_{\widetilde{k_2}} & T_{\widetilde{k_4}} \end{bmatrix}\begin{bmatrix} k_1(\alpha)\delta_1 \\ k_3(\alpha)\delta_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{since } k_1(\alpha) = k_3(\alpha) = 0)\,, \end{aligned}$$

which leads to a contradiction. If instead $p < q$ then the same argument leads to a contradiction. This proves (54).

Now since $p = q$, i.e., $s = 0$, it follows again from (55) and (21) that

$$\begin{bmatrix} \delta_1 \\ \beta\delta_1 \end{bmatrix} \in \text{cl ran}\, H_{A\Omega_2^*} \subseteq \ker\,(I - T_{\widetilde{K}}T_{\widetilde{K}}^*) \quad (\beta \neq 0). \tag{56}$$

We thus have

$$\begin{aligned}
\begin{bmatrix}\delta_1\\ \beta\delta_1\end{bmatrix} &= \begin{bmatrix}T_{\widetilde{k}_1} & T_{\widetilde{k}_3}\\ T_{\widetilde{k}_2} & T_{\widetilde{k}_4}\end{bmatrix}\begin{bmatrix}T_{\overline{\widetilde{k}}_1} & T_{\overline{\widetilde{k}}_2}\\ T_{\overline{\widetilde{k}}_3} & T_{\overline{\widetilde{k}}_4}\end{bmatrix}\begin{bmatrix}\delta_1\\ \beta\delta_1\end{bmatrix}\\
&= \begin{bmatrix}T_{\widetilde{k}_1} & T_{\widetilde{k}_3}\\ T_{\widetilde{k}_2} & T_{\widetilde{k}_4}\end{bmatrix}\begin{bmatrix}(k_1(\alpha)+\beta k_2(\alpha))\delta_1\\ (k_3(\alpha)+\beta k_4(\alpha))\delta_1\end{bmatrix} \qquad (57)\\
&= \begin{bmatrix}\Big(\widetilde{k}_1\big(k_1(\alpha)+\beta k_2(\alpha)\big)+\widetilde{k}_3\big(k_3(\alpha)+\beta k_4(\alpha)\big)\Big)\delta_1\\ \Big(\widetilde{k}_2\big(k_1(\alpha)+\beta k_2(\alpha)\big)+\widetilde{k}_4\big(k_3(\alpha)+\beta k_4(\alpha)\big)\Big)\delta_1\end{bmatrix},
\end{aligned}$$

which can be written as

$$\alpha_1 k_1+\alpha_2 k_3 = 1 \quad\text{and}\quad \alpha_1 k_2+\alpha_2 k_4=\overline{\beta}\,, \tag{58}$$

where $\alpha_1=\overline{k_1(\alpha)+\beta k_2(\alpha)}$ and $\alpha_2=\overline{k_3(\alpha)+\beta k_4(\alpha)}$. From (56) we also have

$$\left|\left|\begin{bmatrix}\delta_1\\ \beta\delta_1\end{bmatrix}\right|\right|_2 = \left|\left|T^*_{\widetilde{K}}\begin{bmatrix}\delta_1\\ \beta\delta_1\end{bmatrix}\right|\right|_2 = \left|\left|\begin{bmatrix}(k_1(\alpha)+\beta k_2(\alpha))\delta_1\\ (k_3(\alpha)+\beta k_4(\alpha))\delta_1\end{bmatrix}\right|\right|_2 = \left|\left|\begin{bmatrix}\overline{\alpha_1}\delta_1\\ \overline{\alpha_2}\delta_1\end{bmatrix}\right|\right|_2, \tag{59}$$

which implies

$$1+|\beta|^2=|\alpha_1|^2+|\alpha_2|^2. \tag{60}$$

Recall that $\varphi_-=b_\alpha\theta_0'\overline{a}$, $\psi_-=b_\alpha\theta_1'\overline{b}$, $\varphi_+=b_\alpha\theta_1'\theta_3\overline{d}$, and $\psi_+=b_\alpha\theta_0'\theta_2\overline{c}$. Thus from (41), we can see that

$$k_1=\theta_2k_1',\ k_2=\theta_1'\theta_3k_2',\ k_3=\theta_0'\theta_2k_3',\ k_4=\theta_3k_4', \tag{61}$$

where $k_i'\in H^\infty$ for $i=1,\cdots,4$. We claim that

$$\theta_2 \text{ and } \theta_3 \text{ are both constant:} \tag{62}$$

indeed, by (58) and (61),

$$\begin{aligned}
\alpha_1k_1+\alpha_2k_3=1 &\Longrightarrow \alpha_1\theta_2k_1'+\alpha_2\theta_0'\theta_2k_3'=1\\
&\Longrightarrow \theta_2(\alpha_1k_1'+\alpha_2\theta_0'k_3')=1\\
&\Longrightarrow \overline{\theta_2}=\alpha_1k_1'+\alpha_2\theta_0'k_3'\in H^\infty\cap\overline{H^\infty}=\mathbb{C}\\
&\Longrightarrow \theta_2 \text{ is constant}
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1k_2+\alpha_2k_4=\overline{\beta} &\Longrightarrow \alpha_1\theta_1'\theta_3k_2'+\alpha_2\theta_3k_4'=\overline{\beta}\\
&\Longrightarrow \theta_3(\alpha_1\theta_1'k_2'+\alpha_2k_4')=\overline{\beta}\\
&\Longrightarrow \overline{\theta_3}=\frac{1}{\overline{\beta}}(\alpha_1\theta_1'k_2'+\alpha_2k_4')\in H^\infty\cap\overline{H^\infty}=\mathbb{C}\quad(\text{since }\beta\neq 0)\\
&\Longrightarrow \theta_3 \text{ is constant}\,,
\end{aligned}$$

which proves (62). Without loss of generality, we may assume that

$$\theta_2=\theta_3=1. \tag{63}$$

We next claim that

$$\theta_0=\theta_1=b_\alpha,\ \ \text{i.e.,}\ \ \theta_0' \text{ and } \theta_1' \text{ are both constant.} \tag{64}$$

If φ and ψ are rational functions having the same number of poles (this hypothesis has not been used until now) then we can see that

$$\text{if } \theta_0' \text{ or } \theta_1' \text{ is constant then both } \theta_0' \text{ and } \theta_1' \text{ are constant:} \tag{65}$$

indeed, since $\varphi_-=\theta_0\overline{a}$ and $\psi_-=\theta_1\overline{b}$ are rational functions, it follows that θ_0 and θ_1 are finite Blaschke products, and hence by assumption, $\deg(\theta_0)=\sharp(\text{poles of }\overline{\varphi_-})=\sharp(\text{poles of }\overline{\psi_-})=\deg(\theta_1)$, giving (65).

Toward (64), and in view of (65), we assume to the contrary that both θ_0' and θ_1' are not constant. Since θ_0' and θ_1' are non-constant finite Blaschke products, there exist $v,w\in\mathbb{D}$ such that $\theta_0'(v)=0=\theta_1'(w)$. But since $k_3=\theta_0'k_3'$ and $k_2=\theta_1'k_2'$, it follows from (58) that

$$k_1(v)=\frac{1}{\alpha_1}\quad\text{and}\quad k_4(w)=\frac{\overline{\beta}}{\alpha_2} \tag{66}$$

(where we note that $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$). Observe that $|k_1(v)| = 1 = |k_4(w)|$: indeed, if $|k_1(v)| < 1$, then $|\alpha_1| > 1$, so that by (60), $|\alpha_2| < |\beta|$, which implies $|k_4(w)| > 1$, which contradicts the fact $||K||_\infty \leq 1$ and if instead $|k_4(w)| < 1$, then similarly we get a contradiction. Since $||k_1||_\infty \leq 1$ and $||k_4||_\infty \leq 1$, it follows from the Maximum Modulus Theorem that k_1 and k_4 are both constant, i.e.,

$$k_1 = \frac{1}{\alpha_1} \quad \text{and} \quad k_4 = \frac{\overline{\beta}}{\alpha_2}. \tag{67}$$

Then from (58), we should have $k_2 = k_3 \equiv 0$, which leads to a contradiction, using (41).

In view of (53) and (64), we can now write

$$\Phi_- = \begin{bmatrix} b_\alpha & b_\alpha \overline{b} \\ b_\alpha \overline{a} & b_\alpha \end{bmatrix} = B^* \Omega_2 \quad \text{(left coprime factorization)},$$

where

$$\Omega_2 := \nu \begin{bmatrix} b_\alpha & -\overline{\gamma} b_\alpha \\ \gamma & 1 \end{bmatrix}.$$

Also, in view of (63) and (64), we can write

$$\begin{aligned} \Phi_+ = \begin{bmatrix} 0 & b_\alpha \overline{d} \\ b_\alpha \overline{c} & 0 \end{bmatrix} &= \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}^* \begin{bmatrix} b_\alpha & 0 \\ 0 & b_\alpha \end{bmatrix} \\ &= A^* \Omega_0 \Omega_2 = A^* \left(\nu \begin{bmatrix} 1 & \overline{\gamma} b_\alpha \\ -\gamma & b_\alpha \end{bmatrix} \right) \left(\nu \begin{bmatrix} b_\alpha & -\overline{\gamma} b_\alpha \\ \gamma & 1 \end{bmatrix} \right), \end{aligned}$$

where $\Omega_0 := \nu \begin{bmatrix} 1 & \overline{\gamma} b_\alpha \\ -\gamma & b_\alpha \end{bmatrix}$ and $\Omega_0 \Omega_2 = b_\alpha I_2$. Then by (20),

$$\Omega_0 H^2_{\mathbb{C}^2} \subseteq \ker\,[T^*_\Phi, T_\Phi], \quad \text{so that} \quad \operatorname{ran}\,[T^*_\Phi, T_\Phi] \subseteq \mathcal{H}_{\Omega_0}. \tag{68}$$

Since $\dim \mathcal{H}_{\Omega_0} = 1$, it follows that $\operatorname{ran}\,[T^*_\Phi, T_\Phi] = \mathcal{H}_{\Omega_0}$ or $\operatorname{ran}\,[T^*_\Phi, T_\Phi] = \{0\}$, i.e., T_Φ is normal. If T_Φ is normal then the same argument as (52) shows that

$$\varphi = \mu\, \overline{b}_\alpha + e^{i\theta} \sqrt{1 + |\mu|^2}\, b_\alpha + \zeta \quad \text{and} \quad \psi = e^{i\,(\pi - 2\arg \mu)} \varphi,$$

where $|\mu| = 1$, $\zeta \in \mathbb{C}$, and $\theta \in [0, 2\pi)$.

Suppose $\operatorname{ran}\,[T^*_\Phi, T_\Phi] = \mathcal{H}_{\Omega_0}$. We now recall a well-known result of B. Morrel ([Mor]; [Con, p.162]). If $T \in \mathcal{B}(\mathcal{H})$ satisfies the following properties: (i) T is hyponormal; (ii) $[T^*, T]$ is rank-one; and (iii) $\ker\,[T^*, T]$ is invariant for T, then $T - \beta$ is quasinormal for some $\beta \in \mathbb{C}$, i.e., $T - \beta$ commutes with $(T - \beta)^*(T - \beta)$. Since T_Φ satisfies the above three properties, we can conclude that $T_{\Phi - \beta}$ is quasinormal for some $\beta \in \mathbb{C}$. Thus, $T^*_{\Phi-\beta}[T^*_{\Phi-\beta}, T_{\Phi-\beta}] = 0$. But since $[T^*_{\Phi-\beta}, T_{\Phi-\beta}] = [T^*_\Phi, T_\Phi]$, it follows that

$$T_{(\Phi^* - \overline{\beta})}[T^*_\Phi, T_\Phi] = 0. \tag{69}$$

Observe that

$$\Omega_0 = \nu \begin{bmatrix} 1 & \overline{\gamma} b_\alpha \\ -\gamma & b_\alpha \end{bmatrix} = \nu \begin{bmatrix} 1 & \overline{\gamma} \\ -\gamma & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & b_\alpha \end{bmatrix},$$

so that

$$\begin{bmatrix} \overline{\gamma}\delta \\ \delta \end{bmatrix} = \begin{bmatrix} 1 & \overline{\gamma} \\ -\gamma & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \delta \end{bmatrix} \in \mathcal{H}_{\Omega_0} \quad \left(\delta := \tfrac{\sqrt{1-|\alpha|^2}}{1-\overline{\alpha} z}\right),$$

which implies that $\begin{bmatrix} \overline{\gamma}\delta \\ \delta \end{bmatrix} \in \operatorname{ran}\,[T^*_\Phi, T_\Phi]$.

On the other hand, since $a, b, c, d \in \mathcal{H}_{z b_\alpha}$, if we choose $\{1, b_\alpha\}$ as a (not necessarily orthogonal) basis for $\mathcal{H}_{z b_\alpha}$, then $\overline{a} b_\alpha$ and $\overline{b} b_\alpha$ are of the form $\xi_1 b_\alpha + \xi_2$ ($\xi_1, \xi_2 \in \mathbb{C}$) and $c \overline{b}_\alpha$ and $d \overline{b}_\alpha$ are of the form $\eta_1 \overline{b}_\alpha + \eta_2$ ($\eta_1, \eta_2 \in \mathbb{C}$). Thus we may assume that T^*_Φ is of the form

$$T^*_\Phi = \begin{bmatrix} T_{b_\alpha} & \overline{b} T_{b_\alpha} + c T_{\overline{b}_\alpha} + c_0 \\ \overline{a} T_{b_\alpha} + d T_{\overline{b}_\alpha} + d_0 & T_{b_\alpha} \end{bmatrix} \quad (a, b, c, d, c_0, d_0 \in \mathbb{C}).$$

Then by (69) we have

$$\begin{bmatrix} T_{b_\alpha} & \overline{b}T_{b_\alpha} + cT_{\overline{b_\alpha}} + c_0 \\ \overline{a}T_{b_\alpha} + dT_{\overline{b_\alpha}} + d_0 & T_{b_\alpha} \end{bmatrix} \begin{bmatrix} \overline{\gamma}\delta \\ \delta \end{bmatrix} = \begin{bmatrix} \overline{\gamma}\overline{\beta}\delta \\ \overline{\beta}\delta \end{bmatrix}.$$

Now recall the case assumption, which gives $\gamma = -b = -\frac{1}{a}$; it follows that

$$\begin{bmatrix} \overline{\gamma}\overline{\beta} \\ \overline{\beta} \end{bmatrix} = \begin{bmatrix} (\overline{\gamma} + \overline{b})b_\alpha + c_0 \\ (1 + \overline{a}\overline{\gamma})b_\alpha + \overline{\gamma}d_0 \end{bmatrix} = \begin{bmatrix} c_0 \\ \overline{\gamma}d_0 \end{bmatrix}. \tag{70}$$

which implies that

$$d_0 = \frac{\overline{\beta}}{\overline{\gamma}} = \frac{1}{\overline{\gamma}^2}c_0 = \overline{a}^2 c_0.$$

On the other hand, a straightforward calculation shows that

$$[T_\Phi^*, T_\Phi] = \begin{bmatrix} A & * \\ * & * \end{bmatrix},$$

where

$$\begin{aligned} A :=& \Big((|c|^2 + |c_0|^2) - (1 + |a|^2 + |d_0|^2)\Big) + \Big(\overline{bc_0} + c_0\overline{c} - \overline{d}d_0 - \overline{d_0}\overline{a}\Big)T_{b_\alpha} \\ &+ \Big(bc_0 + c\overline{c_0} - d\overline{d_0} - d_0 a\Big)T_{\overline{b_\alpha}} + \Big(\overline{bc} - \overline{ad}\Big)T_{b_\alpha^2} + \Big(bc - ad\Big)T_{\overline{b_\alpha^2}} + (1 + |b|^2 - |d|^2)T_{b_\alpha}T_{\overline{b_\alpha}}. \end{aligned} \tag{71}$$

But since $\operatorname{rank} A \le 1$, we have

$$\begin{cases} bc = ad \\ bc_0 + c\overline{c_0} - d\overline{d_0} - d_0 a = 0. \end{cases} \tag{72}$$

Since $ab = 1$ (and hence, $c = a^2 d$) and $\overline{d_0} = a^2\overline{c_0}$, we have $c\overline{c_0} - d\overline{d_0} = a^2 d\overline{c_0} - da^2\overline{c_0} = 0$. Thus by (72),

$$bc_0 = ad_0 = a\overline{a}^2 c_0. \tag{73}$$

If $c_0 \ne 0$, then by (73), $\frac{1}{a} = a\overline{a}^2$, i.e., $|a| = 1$, and in turn $|c_0| = |d_0|$. Also since $ab = 1$, we can write

$$a = e^{i\theta} \quad \text{and} \quad b = e^{-i\theta} \quad \text{for some } \theta \in [0, 2\pi).$$

Since $|c_0| = |d_0|$ and by (71),

$$|c|^2 + |c_0|^2 - (1 + |a|^2 + |d_0|^2) + (1 + |b|^2 - |d|^2) = 0,$$

it follows that $|c| = |d|$. Moreover, a straightforward calculation shows that

$$[T_\Phi^*, T_\Phi] = \begin{bmatrix} |d|^2 - 2 & -2e^{i\theta} \\ -2e^{-i\theta} & |d|^2 - 2 \end{bmatrix} K_0 \quad (\text{where } K_0 := 1 - T_{b_\alpha}T_{\overline{b_\alpha}}).$$

Since $\operatorname{rank}[T_\Phi^*, T_\Phi] = 1$, it follows that $(|d|^2 - 2)^2 - 4 = 0$, i.e., $|d| = 2$. Thus we can write

$$a = e^{i\theta}, \quad b = e^{-2i\theta}a, \quad \overline{d} = 2e^{i\omega}, \quad \overline{c} = e^{-2i\theta}\overline{d} \quad (\text{since } bc = ad).$$

Also since $\overline{c_0} = \overline{a}^2\overline{d_0} = e^{-2i\theta}\overline{d_0}$, it follows that

$$\begin{cases} \varphi = a\overline{b}_\alpha + \overline{d}b_\alpha + \overline{d_0} = e^{i\theta}\overline{b}_\alpha + 2e^{i\omega}b_\alpha + \overline{d_0} \\ \psi = b\overline{b}_\alpha + \overline{c}b_\alpha + \overline{c_0} = e^{-2i\theta}\varphi. \end{cases} \tag{74}$$

In particular, since $\beta = \gamma\overline{d_0} = -\frac{1}{a}\overline{d_0} = -e^{-i\theta}\overline{d_0}$, it follows that $T_\Phi - \beta = T_\Phi + e^{-i\theta}\overline{d_0}$ is quasinormal.

If $c_0 = 0$ then by (70), $\beta = 0$, and hence $d_0 = 0$. In particular, T_Φ is quasinormal. A straightforward calculation shows that

$$[T_\Phi^*, T_\Phi] = \begin{bmatrix} A & -(a + \overline{b})K_0 \\ -(b + \overline{a})K_0 & B \end{bmatrix},$$

where

$$\begin{cases} A := |c|^2 - 1 - |a|^2 + (bc - ad)T_{\overline{b_\alpha^2}} + \overline{(bc - ad)}\,T_{b_\alpha^2} + (1 + |b|^2 - |d|^2)T_{b_\alpha}T_{\overline{b_\alpha}} \\ B := |d|^2 - 1 - |b|^2 + (ad - bc)T_{\overline{b_\alpha^2}} + \overline{(ad - bc)}\,T_{b_\alpha^2} + (1 + |a|^2 - |c|^2)T_{b_\alpha}T_{\overline{b_\alpha}} \\ K_0 := 1 - T_{b_\alpha}T_{\overline{b_\alpha}}. \end{cases}$$

Since rank $[T_\Phi^*, T_\Phi] = 1$, we have

$$\begin{cases} bc = ad \\ |c|^2 - 1 - |a|^2 = |d|^2 - |b|^2 - 1. \end{cases}$$

We thus have

$$[T_\Phi^*, T_\Phi] = \begin{bmatrix} |c|^2 - 1 - |a|^2 & -(a + \overline{b}) \\ -(b + \overline{a}) & |c|^2 - 1 - |a|^2 \end{bmatrix} K_0 \quad (\text{where } K_0 := 1 - T_{b_\alpha} T_{\overline{b}_\alpha}).$$

Thus

$$\begin{aligned} 0 = T_\Phi^*[T_\Phi^*, T_\Phi] &= \begin{bmatrix} T_{b_\alpha} & \overline{b} T_{b_\alpha} + c T_{\overline{b}_\alpha} \\ \overline{a} T_{b_\alpha} + d T_{\overline{b}_\alpha} & T_{b_\alpha} \end{bmatrix} \begin{bmatrix} |c|^2 - 1 - |a|^2 & -(a + \overline{b}) \\ -(b + \overline{a}) & |c|^2 - 1 - |a|^2 \end{bmatrix} K_0 \\ &= \begin{bmatrix} \Big((|c|^2 - 1 - |a|^2) - \overline{b}(b + \overline{a})\Big) T_{b_\alpha} K_0 & * \\ * & \Big(-\overline{a}(a + \overline{b}) + (|c|^2 - 1 - |a|^2)\Big) T_{b_\alpha} K_0 \end{bmatrix}, \end{aligned}$$

which implies

$$\begin{cases} |c|^2 - 1 - |a|^2 - |b|^2 - 1 = 0 \\ |c|^2 - 1 - |a|^2 - |a|^2 - 1 = 0, \end{cases}$$

giving $|a| = |b| = 1$ and in turn $|c| = |d| = 2$. As in (74), we may thus write

$$\begin{cases} \varphi = e^{i\theta} \overline{b}_\alpha + 2 e^{i\omega} b_\alpha \\ \psi = e^{-2i\theta} \varphi. \end{cases}$$

This completes the proof. $\square$

We can say more about the solution of the case (31).

Corollary 4.4. Using the terminology in case (31), assume that either φ or ψ is a rational function having at least two poles. Then both of φ and ψ are rational. Moreover, in this case, either φ or ψ has exactly one pole, say α.

Proof. Suppose either φ or ψ is a rational function having at least two poles. Thus either θ_0' or θ_1' is a nonconstant finite Blaschke product. Without loss of generality we assume that θ_0' is a nonconstant finite Blaschke product. If θ_1' has a nonconstant Blaschke factor then the same argument as in (66) leads to a contradiction. Therefore, for the first assertion, we assume to the contrary that θ_1' is a nonconstant singular inner function. Since θ_0' is a nonconstant finite Blaschke product,

$$\exists\ w \in \mathbb{D} \text{ such that } \theta_0'(w) = 0, \text{ so that by (61), } k_3(w) = 0.$$

Thus by (58), $k_1(w) = \frac{1}{\alpha_1}$. But since $|k_1(w)| < 1$ (if it were not so, then k_1 would be constant, so that $k_3 \equiv 0$, a contradiction from (41)), it follows that $1 < |\alpha_1|$. Thus by (60),

$$|\alpha_2| < |\beta|. \tag{75}$$

On the other hand, since θ_1' is a nonconstant singular inner function, we can see that there exists $\delta \in [0, 2\pi)$ such that θ_1' has nontangential limit 0 at $e^{i\delta}$ (cf. [Ga, Theorem II.6.2]). Thus by (61), k_2 has nontangential limit 0 at $e^{i\delta}$ and in turn, by (58), k_4 has nontangential limit $\frac{\beta}{\alpha_2}$ at $e^{i\delta}$. But since $||k_4||_\infty \le 1$, it follows that $\left|\frac{\beta}{\alpha_2}\right| \le 1$, i.e., $|\beta| \le |\alpha_2|$, which contradicts (75). This proves the first assertion. The second assertion follows at once from the same argument as in (66): in other words, either θ_0' or θ_1' is constant, i.e., $\theta_0 = b_\alpha$ or $\theta_1 = b_\alpha$. This complete the proof. $\square$

Remark 4.5. Due to a technical problem, we omitted a detailed proof for the case B-2 from the proof of [CHL1, Theorem 5.1]. The proof of the case B-2 (with $\alpha = 0$) in the proof of Theorem 4.2 provides the portion of the proof that did not appear in [CHL1]. In particular, Theorem 4.2 incorporates an extension of a corrected version of [CHL1, Theorem 5.1], in which the exceptional case (31) was omitted.

References

[Ab] M.B. Abrahamse, *Subnormal Toeplitz operators and functions of bounded type*, Duke Math. J. **43** (1976), 597–604.

[AIW] I. Amemiya, T. Ito, and T.K. Wong, *On quasinormal Toeplitz operators*, Proc. Amer. Math. Soc. **50** (1975), 254–258.

[BS] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*, Springer, Berlin-Heidelberg, 2006.

[BH] A. Brown and P.R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/1964), 89–102.

[Con] J.B. Conway, The Theory of Subnormal Operators, *Math. Surveys and Monographs*, **36**(1991), Amer. Math. Soc. Providence

[Co1] C. Cowen, *More subnormal Toeplitz operators*, J. Reine Angew. Math. **367**(1986), 215–219.

[Co2] C. Cowen, *Hyponormal and subnormal Toeplitz operators*, in: J.B. Conway and B.B. Morrel (Eds.), Survey of Soem Recent Results in Operator Theory, I, in: Pitman Res. Notes Math., vol. 171, Longman, 1988, pp. 155-167.

[Co3] C. Cowen, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103**(1988), 809–812.

[CoL] C. Cowen and J. Long, *Some subnormal Toeplitz operators*, J. Reine Angew. Math. **351**(1984), 216–220.

[CHL1] R.E. Curto, I.S. Hwang and W.Y. Lee, *Hyponormality and subnormality of block Toeplitz operators*, Adv. Math. **230**(2012), 2094-2151.

[CHL2] R.E. Curto, I.S. Hwang and W.Y. Lee, *Which subnormal Toeplitz operators are either normal or analytic ?*, J. Funct. Anal. **263(8)**(2012), 2333-2354.

[CuL1] R.E. Curto and W.Y. Lee, *Joint hyponormality of Toeplitz pairs*, Mem. Amer. Math. Soc., vol. 712, Amer. Math. Soc., Providence, RI, 2001.

[CuL2] R.E. Curto and W.Y. Lee, *Subnormality and k-hyponormality of Toeplitz operators: A brief survey and open question*, in:Operator Theory and Banach Algebras, Theta, Bucharest, 2003, pp. 73-81.

[Do1] R.G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.

[Do2] R.G. Douglas, *Banach Algebra Techniques in the Theory of Toeplitz Operators*, CBMS Reg. Conf. Ser. Math., vol. 15, Amer. Math. Soc., Providence, RI, 1973.

[FL] D.R. Farenick and W.Y. Lee, *Hyponormality and spectra of Toeplitz operators*, Trans. Amer. Math. Soc. **348**(1996), 4153–4174.

[FF] C. Foiaş and A. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, Operator Theory: Adv. Appl. vol 44, Birkhäuser, Boston, 1993.

[Ga] J.B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[GGK] I. Gohberg, S. Goldberg, and M.A. Kaashoek, *Classes of Linear Operators, Vol II*, Basel, Birkhauser, 1993.

[Gu1] C. Gu, *A generalization of Cowen's characterization of hyponormal Toeplitz operators*, J. Funct. Anal. **124**(1994), 135–148.

[Gu2] C. Gu, *On a class of jointly hyponormal Toeplitz operators*, Trans. Amer. Math. Soc.**354**(2002), 3275–3298.

[GHR] C. Gu, J. Hendricks and D. Rutherford, *Hyponormality of block Toeplitz operators*, Pacific J. Math. **223** (2006), 95–111.

[GS] C. Gu and J.E. Shapiro, *Kernels of Hankel operators and hyponormality of Toeplitz operators*, Math. Ann. **319**(2001), 553–572.

[Hal1] P. R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76**(1970), 887–933.

[Hal2] P. R. Halmos, *Ten years in Hilbert space*, Integral Equations Operator Theory **2**(1979), 529–564.

[HKL1] I. S. Hwang, I. H. Kim and W.Y. Lee, *Hyponormality of Toeplitz operators with polynomial symbols*, Math. Ann. **313(2)** (1999), 247-261.

[HKL2] I. S. Hwang, I. H. Kim and W.Y. Lee, *Hyponormality of Toeplitz operators with polynomial symbols: An extremal case*, Math. Nach. **231** (2001), 25-38.

[HL1] I. S. Hwang and W.Y. Lee, *Hyponormality of trigonometric Toeplitz operators*, Trans. Amer. Math. Soc. **354** (2002), 2461-2474.

[HL2] I. S. Hwang and W.Y. Lee, *Hyponormality of Toeplitz operators with rational symbols*, Math. Ann. **335**(2006), 405–414.

[HL3] I. S. Hwang and W.Y. Lee, *Hyponormal Toeplitz operators with rational symbols*, J. Operator Theory **56**(2006), 47–58.

[Le] W. Y. Lee, *Cowen sets for Toeplitz operators with finite rank selfcommutators*, J. Operator Theory **54(2)**(2005), 301-307.

[Mor] B.B. Morrel, A decomposition for some operators, *Indiana Univ. Math. J.*, **23**(1973), 497-511.

[NT] T. Nakazi and K. Takahashi, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces*, Trans. Amer. Math. Soc. **338**(1993), 753–769.

[Ni] N. K. Nikolskii, *Treatise on the Shift Operator*, Springer, New York, 1986.

[Pe] V. V. Peller, *Hankel Operators and Their Applications*, Springer, New York, 2003.

[Po] V.P. Potapov, *On the multiplicative structure of J-nonexpansive matrix functions*, Tr. Mosk. Mat. Obs. (1955), 125-236 (in Russian); English trasl. in: Amer. Math. Soc. Transl. (2) 15(1966), 131-243.

[Ya] D.V. Yakubovich, *Real separated algebraic curves, quadrature domains, Ahlfors type functions and operator theory*, J. Funct. Anal. **236**(2006), 25-58.

[Zhu] K. Zhu, *Hyponormal Toeplitz operators with polynomial symbols*, Integral Equations Operator Theory **21**(1996), 376–381

Raúl E. Curto
Department of Mathematics, University of Iowa, Iowa City, IA 52242, U.S.A.
e-mail: `raul-curto@uiowa.edu`

In Sung Hwang
Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea
e-mail: `ihwang@skku.edu`

Woo Young Lee
Department of Mathematics, Seoul National University, Seoul 151-742, Korea
e-mail: `wylee@snu.ac.kr`